Local Cohomology and Matlis duality – Table of contents

0 Introduction . . . 2

1 Motivation and General Results . . . 6

1.1 Motivation . . . 6

1.2 Conjecture (*) on the structure of $\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R)))$ . . . 10

1.3 Regular sequences on $D(\mathrm{H}^h_I(R))$ are well-behaved in some sense . . . 13

1.4 Comparison of two Matlis Duals . . . 15

2 Associated primes – a constructive approach . . . 19

3 Associated primes – the characteristic-free approach . . . 26

3.1 Characteristic-free versions of some results . . . 26

3.2 On the set $\mathrm{Ass}_R(D(\mathrm{H}^{\dim(R)-1}_I(R)))$ . . . 28

4 The regular case and how to reduce to it . . . 35

4.1 Reductions to the regular case . . . 35

4.2 Results in the general case, i. e. h is arbitrary . . . 36

4.3 The case $h = \dim(R) - 2$, i. e. the set $\mathrm{Ass}_R(D(\mathrm{H}^{n-2}_{(X_1,\ldots,X_{n-2})R}(k[[X_1,\ldots,X_n]])))$ . . . 38

5 On the meaning of a small arithmetic rank of a given ideal . . . 43

5.1 An Example . . . 43

5.2 Criteria for $\mathrm{ara}(I) \leq 1$ and $\mathrm{ara}(I) \leq 2$ . . . 44

5.3 Differences between the local and the graded case . . . 48

6 Applications . . . 50

6.1 Hartshorne-Lichtenbaum vanishing . . . 50

6.2 Generalization of an example of Hartshorne . . . 52

6.3 A necessary condition for set-theoretic complete intersections . . . 54

6.4 A generalization of local duality . . . 55

7 Further Topics . . . 57

7.1 Local Cohomology of formal schemes . . . 57

7.2 $D(\mathrm{H}^i_I(R))$ has a natural D-module structure . . . 57

7.3 The zeroth Bass number of $D(\mathrm{H}^i_I(R))$ (w. r. t. the zero ideal) is not finite in general . . . 58

7.4 On the module $\mathrm{H}^h_I(D(\mathrm{H}^h_I(R)))$ . . . 61

8 Attached prime ideals and local homology . . . 65

8.1 Attached prime ideals – basics . . . 65

8.2 Attached prime ideals – results . . . 68

8.3 Local homology and a necessary condition for Cohen-Macaulayness . . . 71

8.4 Local homology and Cohen-Macaulayfications . . . 74

References . . . 76

Summary in German (deutsche Zusammenfassung) . . . 78

Local Cohomology and Matlis duality

0 Introduction

In algebraic geometry, a (set-theoretic) complete intersection is a variety Y (say, in affine or projective space over a field) that can be cut out be $\mathrm{codim}(Y)$ many equations. For example, every curve in affine n-space over a field of positive characteristic is a set-theoretic complete intersection (see [CN]). On the other hand, many questions on complete intersections are still open: Is every closed point in $\mathbf{P}^2_{\mathbf{Q}}$ (projective 2-space over $\mathbf{Q}$, the rationals) a set-theoretic complete intersection? Is every irreducible curve in $\mathbf{A}^3_{\mathbf{C}}$ (affine 3-space over $\mathbf{C}$, the complex numbers) a set-theoretic complete intersection? See [Ly2] for a survey on these and other questions.

Here is another example: Over an algebraically closed field k, let $C_d \subseteq \mathbf{P}^3_k$ be the curve parameterized by

$$(u^d : u^{d-1}v : uv^{d-1} : v^d)$$

(for $(u : d) \in \mathbf{P}^1_k$). Hartshorne has shown (see [Ha2, Theorem.*]) that, in positive characteristic, every curve C_d is a set-theoretic complete intersection. In characteristic zero, the question is open. It is even unknown if C_4 is a set-theoretic complete intersection or not. An obvious obstruction for C_4 to be a set-theoretic complete intersection would be $\mathrm{H}^3_I(R) \neq 0$ ($I \subseteq R = k[X_0, X_1, X_2, X_3]$ the vanishing ideal of $C_4 \subseteq \mathbf{P}^3_k$), but, as is well-known, one has

$$\mathrm{H}^3_I(R) = 0 \quad .$$

Thus, if we define the so-called arithmetic rank of I,

$$\mathrm{ara}(I) := \min\{l \in \mathbf{N} | \exists r_1, \ldots, r_l \in R : \sqrt{I} = \sqrt{(r_1, \ldots, r_l)R}\} \quad ,$$

it seems that (non-)vanishing of the modules $\mathrm{H}^i_I(R)$ does not carry enough information to determine $\mathrm{ara}(I)$ (because our example 5.1 shows that this can really happen in the sense that $\mathrm{cd}(I) < \mathrm{ara}(I)$, where cd is the (local) cohomological dimension of I).

It is interesting that, although the vanishing of $\mathrm{H}^3_I(R)$ does not seem to help in the case of C_4, the Matlis dual $D(\mathrm{H}^2_I(R))$ (note that D will stand for the Matlis dual functor, also see the end of this introduction for more notation) of the module $\mathrm{H}^2_I(R)$ "knows" whether we have a set-theoretic complete intersection or not, in the following sense (take $h = 2$):

(1.1.4 Corollary)
Let $(R, \mathfrak{m})$ be a noetherian local ring, I a proper ideal of R, $h \in \mathbf{N}$ and $\underline{f} = f_1, \ldots, f_h \in I$ an R-regular sequence. The following statements are equivalent:
(i) $\sqrt{\underline{f}R} = \sqrt{I}$, i. e. I is – up to radical – the set-theoretic complete intersection ideal $\underline{f}R$; in particular, it is a set-theoretic complete intersection ideal itself.
(ii) $\mathrm{H}^l_I(R) = 0$ for every $l > h$ and the sequence $\underline{f}$ is quasi-regular on $D(\mathrm{H}^h_I(R))$.
(iii) $\mathrm{H}^l_I(R) = 0$ for every $l > h$ and the sequence $\underline{f}$ is regular on $D(\mathrm{H}^h_I(R))$.

This result gives motivation to study modules of the form $D(\mathrm{H}^l_I(R))$, in particular its associated primes (as they determine which elements operate injectively on $D(\mathrm{H}^l_I(R))$). Modules of the form $D(\mathrm{H}^l_I(R))$ and their associated prime ideals have been studied in [H2], [H3], [H4], [H5], [HS1] and [HS2].

Main results

In the sequel we will list the main results of this work. In this context we would like to remark that conjecture (*) (1.2.2) is a central theme of this work. We also remark that many of the results listed below (e. g. (1.2.1), (2.7) (i), (3.1.3) (ii) and (iii), (3.2.7), (4.1.2), but also (8.2.6) (iii) (ζ)) give evidence for it, though we are not able to prove conjecture (*).

We also note that the results in this work lead to various applications. These applications are collected in section 6, they are not listed here.

(1.2.1 Remark)
Let $(R, \mathfrak{m})$ be a noetherian local ring and $\underline{x} = x_1, \ldots, x_h$ a sequence in R. Then one has

$$\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \mathrm{H}^h_{(x_1,\ldots,x_h)R}(R/\mathfrak{p}) \neq 0\} \quad .$$

Though easy, remark 1.2.1 is crucial for many proofs in this work; it seems reasonable to conjecture:

(1.2.2 Conjecture)
If $(R, \mathfrak{m})$ is a noetherian local ring, $h > 0$ and $x_1, \ldots, x_h$ are elements of R,

$$(*) \qquad \mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) = \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \mathrm{H}^h_{(x_1,\ldots,x_h)R}(R/\mathfrak{p}) \neq 0\} \quad .$$

Besides remark 1.2.1, there is more evidence for conjecture (*), e. g.:

(3.1.3 Theorem, statements (ii) and (iii))
Let $(R, \mathfrak{m})$ be a noetherian local ring, $\underline{x} = x_1, \ldots, x_m$ a sequence in R and M a finitely generated R-module. Then

$$\{\mathfrak{p} \in \mathrm{Supp}_R(M) | x_1, \ldots, x_m \text{ is part of a system of parameters of } R/\mathfrak{p}\} \subseteq \mathrm{Ass}_R(D(\mathrm{H}^m_{(x_1,\ldots,x_m)R}(M)))$$

holds. Now, if we assume furthermore that R is a domain and $\underline{x}$ is part of a system of parameters of R, we have $\{0\} \in \mathrm{Ass}_R(D := D(\mathrm{H}^m_{\underline{x}R}(R)))$. Therefore, it is natural to ask for the zeroth Bass number of D with respect to the zero ideal. We will see that, in general, this number is not finite (theorem 7.3.2). In the special case $m = 1$ we can completely compute the associated prime ideals: Namely, for every $x \in R$, one has

$$\mathrm{Ass}_R(D(\mathrm{H}^1_{xR}(R))) = \mathrm{Spec}(R) \setminus \mathcal{V}(x)$$

($\mathcal{V}(x)$ is the set of all prime ideals of R containing x). In particular, the set

$$\mathrm{Ass}_R(D(\mathrm{H}^m_{(x_1,\ldots,x_m)R}(R)))$$

is, in general, infinite. Here is further evidence for conjecture (*):

(3.2.7 Theorem)
Let $(R, \mathfrak{m})$ be a d-dimensional local complete ring and $J \subseteq R$ an ideal such that $\dim(R/J) = 1$ and $\mathrm{H}^d_J(R) = 0$. Then

$$\mathrm{Ass}_R(D(\mathrm{H}^{d-1}_J(R)) = \{P \in \mathrm{Spec}(R) |\, \dim(R/P) = d-1, \dim(R/(P+J)) = 0\} \cup \mathrm{Assh}(R)$$

holds. Here $\mathrm{Assh}(R)$ denotes the set of all associated prime ideals of R of highest dimension. Further evidence for (*) can be found in section 8.2 in connection with attached primes (see 8.2.6 (iii) (ζ) for details).

We continue our list of main results on Matlis duals of local cohomology modules:

(4.1.2 Theorem)
Let $(R, \mathfrak{m})$ be a noetherian local complete ring with coefficient field $k \subseteq R$, $l \in \mathbb{N}^+$ and $x_1, \ldots, x_l \in R$ a part of a system of parameters of R. Set $I := (x_1, \ldots, x_l)R$. Let $x_{l+1}, \ldots, x_d \in R$ be such that $x_1, \ldots, x_d$ is a system of parameters of R. Denote by R_0 the (regular) subring $k[[x_1, \ldots, x_d]]$ of R. Then if $\mathrm{Ass}_{R_0}(D(\mathrm{H}^l_{(x_1,\ldots,x_l)R_0}(R_0)))$ is stable under generalization, $\mathrm{Ass}_R(D(\mathrm{H}^l_I(R)))$ is also stable under generalization.

(A set X of prime ideals of a ring is stable under generalization, if $\mathfrak{p} \in X$ implies $\mathfrak{p}_0 \in X$ for every $\mathfrak{p}_0 \subseteq \mathfrak{p}$.) Clearly, 4.1.2 can be helpful when we want to reduce from a general (complete) to a regular (complete) case.

The next result shows that the question when $\mathrm{H}^{\dim(R)-1}_I(R)$ is zero (for an ideal I in a local regular ring R) is related to the question which prime ideals are associated to the Matlis dual of a certain local cohomology module:

(4.3.1 Corollary)
Let R_0 be a noetherian local complete equicharacteristic ring, let $\dim(R_0) = n - 1$, $k \subseteq R_0$ a coefficient field of R_0. Let $x_1, \ldots, x_n$ be elements of R_0 such that $\sqrt{(x_1, \ldots, x_n)R_0} = \mathfrak{m}_0$. Set $I_0 := (x_1, \ldots, x_{n-2})R_0$. Let $R := k[[X_1, \ldots, X_n]]$ be a power series algebra over k in the variables $X_1, \ldots, X_n$, $I := (X_1, \ldots, X_{n-2})R$. Then the k-algebra homomorphism $R \to R_0$ determined by $X_i \mapsto x_i$ $(i = 1, \ldots n)$ induces a module-finite ring map $\iota : R/fR \to R_0$ for some prime element $f \in R$. Furthermore, suppose that R_0 is regular and $\mathrm{height}(I_0) < h$; then we have

$$fR \in \mathrm{Ass}_R(D(\mathrm{H}^h_I(R))) \iff \mathrm{H}^{n-2}_{I_0}(R_0) \neq 0 \quad .$$

In this case, fR is a maximal element of $\mathrm{Ass}_R(D(\mathrm{H}^h_I(R)))$. By [HL, Theorem 2.9] the latter holds if and only if $\dim(R_0/I_0) \geq 2$ and $\mathrm{Spec}(\overline{R_0}/I_0\overline{R_0}) \setminus \{\mathfrak{m}_0(\overline{R_0}/I_0\overline{R_0})\}$ is connected, where $\overline{R_0}$ is defined as the completion of the strict henselization of R_0; this means that $\overline{R_0}$ is obtained from R_0 by replacing the coefficient field k by its separable closure in any fixed algebraic closure of k.

It was shown in [Ly1, Example 2.1. (iv)] that every local cohomology module $\mathrm{H}^i_I(R)$ has a natural D-module structure, where

$$D := D(R, k) \subseteq \mathrm{End}_k(R)$$

is the subring generated by all k-linear derivations (from R to R) and the multiplications by elements of R (here $k \subseteq R$ is any subring). We show in section 7.2 that, at least if $R = k[[X_1, \ldots, X_n]]$ is a formal power series ring over k, the Matlis dual

$$D(\mathrm{H}^i_I(R))$$

has a canonical D-module structure, too (for every ideal $I \subseteq R$); furthermore, we will see that, with respect to this D-module structure, $D(\mathrm{H}^i_I(R))$ is not finitely generated, in general; in particular, it is not holonomic (see [Bj, in particular sections 1 and 3] for the notion of holonomic D-modules).

We will use the D-module structure on $D(\mathrm{H}^i_I(R))$ to show

(7.4.1 and 7.4.2 Theorems (special cases))
Let $(R, \mathfrak{m})$ be a noetherian local complete regular ring of equicharacteristic zero, $I \subseteq R$ an ideal of height $h \geq 1$ such that $\mathrm{H}^l_I(R) = 0$ for every $l > h$, and $\underline{x} = x_1, \ldots, x_h$ an R-regular sequence in I; then, in general,

$$\mathrm{H}^h_I(D(\mathrm{H}^h_I(R)))$$

is either $\mathrm{E}_R(R/\mathfrak{m})$ or zero; if we assume

$$I = (x_1, \ldots, x_h)R$$

in addition, we have

$$\mathrm{H}^h_I(D(\mathrm{H}^h_I(R))) = \mathrm{E}_R(R/\mathfrak{m}) \quad .$$

Further main results of this work are contained in section 6, in which we collect various applications of our theory, namely new proofs for Hartshorne-Lichtenbaum vanishing (6.1), a generalization of an example of a non-artinian but zero-dimensional local cohomology module (the original example, which is more special, is from Hartshorne) (6.2), a new necessary condition for an ideal to be a set-theoretic complete intersection ideal (6.3) and a generalization of local duality (6.4).

Notation

If I is an ideal of a ring R and M is an R-module, we denote by $\mathrm{H}^l_I(M)$ the l-th local cohomology of M supported in I; material on local cohomology can e. g. be found in [BH], [BS], [Gr] and [Hu]. If $(R, \mathfrak{m})$ is a noetherian local ring, $\mathrm{E}_R(R/\mathfrak{m})$ stands for any (fixed) R-injective hull of the R-module $R/\mathfrak{m}$; see, for example, [BH] and [Ms] for more on injective modules. Finally, D is the Matlis dual functor from the category of R-modules to itself, i. e.

$$D(M) := \mathrm{Hom}_R(M, \mathrm{E}_R(R/\mathfrak{m}))$$

for every R-module M. The term "Matlis dual of M" will always stand for $D(M)$ (and therefore, will only be used over a local ring $(R, \mathfrak{m})$). Sometimes we will write D_R instead of D to avoid misunderstandings. References for general facts from commutative algebra are [Ei], [Ma].

Acknowledgement

I thank Jürgen Stückrad and Gennady Lyubeznik for many helpful discussions.

1 Motivation and General Results

1.1 Motivation

Let I be an ideal of a noetherian ring R.

$$\text{ara}(I) := \min\{l \in \mathbb{N} | \exists x_1, \dots, x_l \in I : \sqrt{I} = \sqrt{(x_1, \dots, x_l)R}\}$$

denotes the arithmetic rank of I. Geometrically, it is the (minimal) number of equations needed to cut out a given algebraic set (say, in an affine space). It is well-known (and follows by using Čech-cohomology) that one has

$$\mathrm{H}^l_I(R) = 0 \quad (l > \text{ara}(I)) \quad .$$

But, conversely, it is in general not true that $\text{ara}(I)$ is determined by these vanishing conditions, see Example 5.1 for a counterexample. Assume that I is generated up to radical by a regular sequence $\underline{f} = f_1, \dots, f_h$ in R. Then $\underline{f}$ is also a regular sequence on $D(\mathrm{H}^h_I(R))$ (this follows from theorem 1.1.2 resp. corollary 1.1.4 below, see definition 1.1.1 below for a definition of regular sequences in this context). It is an interesting fact that the reversed statement also holds: If $\underline{f}$ is a $D(\mathrm{H}^h_I(R))$-regular sequence then

$$\sqrt{I} = \sqrt{\underline{f}}$$

holds. This fact is one of the main motivations for the study of Matlis duals of local cohomology modules (see theorem 1.1.3 resp. corollary 1.1.4 for details and the precise statement).

1.1.1 Definition

Let R be a ring, M an R-module, $h \in \mathbb{N}$ and $\underline{f} = f_1, \dots, f_h$ a sequence of elements of R. $\underline{f}$ is called a quasi-regular sequence on M if multiplication by f_i is injective on $M/(f_1, \dots, f_{i-1})M$ for every $i = 1, \dots, h$. $\underline{f}$ is called a regular sequence on M if $\underline{f}$ is quasi-regular on M and $M/\underline{f}M \neq 0$ holds, in addition.

Before we show the statements on regular sequences mentioned in the introduction of this section (corollary 1.1.4), we prove something slightly more general (namely theorems 1.1.2 and 1.1.3); corollary 1.1.4 then simply combines the most interesting special cases from these two theorems.

1.1.2 Theorem

Let $(R, \mathfrak{m})$ be a noetherian local ring, I an ideal of R, $h \geq 1$ and $\underline{f} = f_1, \dots, f_h \in I$ a sequence of elements such that $\sqrt{\underline{f}R} = \sqrt{I}$ and

$$\mathrm{H}^{h-1-l}_I(R/(f_1, \dots, f_l)R) = 0 \quad (l = 0, \dots, h-3)$$

hold (of course, for $h \leq 2$, this condition is void). Then $\underline{f}$ is a quasi-regular sequence on $D(\mathrm{H}^h_I(R))$.
Proof:
By induction on h: $h = 1$: the functor H^1_I is right-exact because $\mathrm{H}^2_I = \mathrm{H}^2_{f_1R} = 0$. Hence the exact sequence

$$R \xrightarrow{f_1} R \to R/f_1R \to 0$$

induces an exact sequence

$$\mathrm{H}^1_I(R) \xrightarrow{f_1} \mathrm{H}^1_I(R) \to \mathrm{H}^1_I(R/f_1R) = \mathrm{H}^1_{f_1R}(R/f_1R) = 0$$

(here f_1 stands for multiplication by f_1 on R resp. $\mathrm{H}^1_I(R)$). Application of D to the last sequence yields injectivity of f_1 on $D(\mathrm{H}^1_I(R))$.
$h = 2$: We have $\mathrm{H}^2_I(R/f_1R) = \mathrm{H}^2_{(f_1,f_2)R}(R/f_1R) = 0$ This implies both $\mathrm{H}^l_I(M) = 0$ for every $l \geq 2$ and every R/f_1R-module M and the fact that f_1 operates injectively on $D(\mathrm{H}^2_I(R))$. Now the exact sequence

$$0 \to (0 :_R f_1) \to R \xrightarrow{\alpha} f_1R \to 0$$

(where α is induced by multiplication by f_1) induces an exact sequence

$$\mathrm{H}^2_I((0 :_R f_1)) \to \mathrm{H}^2_I(R) \xrightarrow{\mathrm{H}^2_I(\alpha)} \mathrm{H}^2_I(f_1R) \to 0 \ .$$

But $(0 :_R f_1R)$ is an R/f_1R-module and so $\mathrm{H}^2_I((0 :_R f_1R)) = 0$, showing that $\mathrm{H}^2_I(\alpha)$ is an isomorphism. On the other hand the exact sequence

$$0 \to f_1R \xrightarrow{\beta} R \to R/f_1R \to 0$$

(where β is an inclusion map) induces an exact sequence

$$\mathrm{H}^1_I(R/f_1R) \to \mathrm{H}^2_I(f_1R) \xrightarrow{\mathrm{H}^2_I(\beta)} \mathrm{H}^2_I(R) \to 0 \ ,$$

which shows the existence of a natural epimorphism

$$\begin{aligned} \mathrm{H}^1_I(R/f_1R) &\to \ker(\mathrm{H}^2_I(\beta)) \\ &\cong \ker(\mathrm{H}^2_I(\beta) \circ \mathrm{H}^2_I(\alpha)) \\ &= \ker(\mathrm{H}^2_I(\beta \circ \alpha)) \\ &= \ker(f_1) \quad , \end{aligned}$$

where f_1 denotes multiplication by f_1 on $\mathrm{H}^2_I(R)$. This means that we have a surjection

$$\mathrm{H}^1_I(R/f_1R) \to \mathrm{Hom}_R(R/f_1R, \mathrm{H}^2_I(R))$$

and hence an injection

$$D(\mathrm{H}^2_I(R))/f_1D(\mathrm{H}^2_I(R)) = D(\mathrm{Hom}_R(R/f_1R, \mathrm{H}^2_I(R))) \to D(\mathrm{H}^1_I(R/f_1R)) \quad .$$

Note that the first equality follows formally from the exactness of D; note also that it does not make any difference if one takes the last Matlis dual with respect to R or R/f_1R. For this reason the case $h = 1$ shows that f_2 operates injectively on $D(\mathrm{H}^1_I(R/f_1R))$ and thus also on $D(\mathrm{H}^2_I(R))/f_1D(\mathrm{H}^2_I(R))$.
Now we consider the general case $h \geq 3$: Similar to the case $h = 2$ we see that $\mathrm{H}^l_I(M) = 0$ for every $l \geq h$ and every R/f_1R-module M and that f_1 operates injectively on $D(\mathrm{H}^h_I(R))$. The short exact sequence

$$0 \to (0 :_R f_1) \to R \xrightarrow{\alpha} f_1R \to 0$$

(where, again, α is induced by multiplication by f_1 on R) induces an exact sequence

$$\mathrm{H}^h_I((0 :_R f_1)) \to \mathrm{H}^h_I(R) \xrightarrow{\mathrm{H}^h_I(\alpha)} \mathrm{H}^h_I(f_1R) \to 0 \ .$$

But $(0:_R f_1)$ is an R/f_1R-module and therefore $\mathrm{H}^h_I((0:_R f_1)) = 0$, showing that $\mathrm{H}^h_I(\alpha)$ is an isomorphism. On the other hand the short exact sequence

$$0 \to f_1R \xrightarrow{\beta} R \to R/f_1R \to 0$$

(where β is an inclusion map) induces an exact sequence

$$0 = \mathrm{H}^{h-1}_I(R) \to \mathrm{H}^{h-1}_I(R/f_1R) \to \mathrm{H}^h_I(f_1R) \xrightarrow{\mathrm{H}^h_I(\beta)} \mathrm{H}^h_I(R) \to 0$$

(here we use the fact that $h \geq 3$ and therefore $\mathrm{H}^{h-1}_I(R) = 0$). We conclude

$$\mathrm{H}^{h-1}_I(R/f_1R) = \ker(\mathrm{H}^h_I(\beta)) \cong \ker(\mathrm{H}^h_I(\beta) \circ \mathrm{H}^h_I(\alpha)) = \mathrm{Hom}_R(R/f_1R, \mathrm{H}^h_I(R))$$

and, by Matlis duality,

$$D(\mathrm{H}^{h-1}_I(R/f_1R)) = D(\mathrm{Hom}_R(R/f_1R, \mathrm{H}^h_I(R))) = D(\mathrm{H}^h_I(R))/f_1D(\mathrm{H}^h_I(R)) \ .$$

Because of our hypothesis, we can apply the induction hypothesis (to the ring R/f_1R) which says that $f_2, \dots, f_h$ is a quasi-regular sequence on $D(\mathrm{H}^{h-1}_I(R/f_1R))$; thus, by the last formula, $\underline{f}$ is a quasi-regular sequence on $D(\mathrm{H}^h_I(R))$.

1.1.3 Theorem

Let I be an ideal of a noetherian local ring $(R, \mathfrak{m})$, $h \geq 1$ and $f_1, \dots, f_h \in I$ be such that

$$\mathrm{H}^l_I(R) = 0 \quad (l > h)$$

and

$$\mathrm{H}^{h-1-l}_I(R/(f_1, \dots, f_l)R) = 0 \quad (l = 0, \dots, h-2)$$

hold (of course, for $h < 2$, this condition is void) and such that the sequence $\underline{f} = f_1, \dots, f_h$ is quasi-regular on $D(\mathrm{H}^h_I(R))$. Then $\sqrt{I} = \sqrt{(f_1, \dots, f_h)R}$ holds.

Proof:

By induction on h: $h = 1$: By our hypothesis, the functor H^1_I is right-exact. Therefore the exact sequence

$$R \xrightarrow{f_1} R \to R/f_1R \to 0$$

induces an exact sequence

$$\mathrm{H}^1_I(R) \xrightarrow{f_1} \mathrm{H}^1_I(R) \to \mathrm{H}^1_I(R/f_1R) \to 0 \quad ,$$

where f_1 stands for multiplication by f_1 on R resp. on $\mathrm{H}^1_I(R)$. But multiplication by f_1 is injective on $D(\mathrm{H}^1_I(R))$ and so we get $\mathrm{H}^1_I(R/f_1R) = 0$; by our hypothesis, we have $\mathrm{H}^l_I(R/f_1R) = 0$ for every $l \geq 1$. It is well-known that the latter condition is equivalent to $\mathrm{H}^l_I(R/\mathfrak{p}) = 0$ for every $l \geq 1$ and every prime ideal $\mathfrak{p}$ of

$$\mathcal{V}(f_1R) := \{\mathfrak{p} \in \mathrm{Spec}(R) | f_1R \subseteq \mathfrak{p}\} \quad .$$

Thus, we must have $I \subseteq \sqrt{f_1R}$ and, therefore, $\sqrt{I} = \sqrt{f_1R}$.

$h \geq 2$: Similar to the case $h = 1$ we see that $\mathrm{H}^h_I(R/f_1R) = 0$ holds. By our hypothesis, we get $\mathrm{H}^l_I(M) = 0$ for every $l \geq h$ and every R/f_1R-module M. The short exact sequence

$$0 \to (0:_R f_1) \to R \xrightarrow{\alpha} f_1R \to 0$$

induces an exact sequence

$$\mathrm{H}^h_I((0:_R f_1)) \to \mathrm{H}^h_I(R) \overset{\mathrm{H}^h_I(\alpha)}{\to} \mathrm{H}^h_I(f_1R) \to 0 \quad .$$

But $\mathrm{H}^h_I((0:_R f_1)) = 0$, because $(0:_R f_1)$ is an R/f_1R-module. Thus $\mathrm{H}^h_I(\alpha)$ is an isomorphism. Now the short exact sequence

$$0 \to f_1R \overset{\beta}{\to} R \to R/f_1R \to 0$$

(where β is an inclusion map) induces an exact sequence

$$0 = \mathrm{H}^{h-1}_I(R) \to \mathrm{H}^{h-1}_I(R/f_1R) \to \mathrm{H}^h_I(f_1R) \overset{\mathrm{H}^h_I(\beta)}{\to} \mathrm{H}^h_I(R) \to 0 \quad ,$$

from which we conclude

$$\mathrm{H}^{h-1}_I(R/f_1R) = \ker(\mathrm{H}^h_I(\beta)) \cong \ker(\mathrm{H}^h_I(\beta) \circ \mathrm{H}^h_I(\alpha)) = \mathrm{Hom}_R(R/f_1R, \mathrm{H}^h_I(R)) \quad .$$

Here we used the facts the $\mathrm{H}^h_I(\alpha)$ is an isomorphism and that $\beta \circ \alpha$ is multiplication by f_1 on R. Application of the functor D shows

$$D(\mathrm{H}^{h-1}_I(R/f_1R)) = D(\mathrm{H}^h_I(R))/f_1 D(\mathrm{H}^h_I(R)) \quad .$$

Note that, again, it is irrelevant whether we take the first functor D here with respect to R or with respect to R/f_1R and so our induction hypothesis (applied to R/f_1R) implies that $f_2, \ldots, f_h$ is a quasi-regular sequence on $D(\mathrm{H}^{h-1}_I(R/f_1R))$ and that

$$\sqrt{I(R/f_1R)} = \sqrt{(f_2, \ldots, f_h) \cdot (R/f_1R)}$$

holds. The statement $\sqrt{I} = \sqrt{(f_1, \ldots, f_h)R}$ follows.

Now it is easy to specialize to the following statement:

1.1.4 Corollary

Let $(R, \mathfrak{m})$ be a noetherian local ring, I a proper ideal of R, $h \in \mathbb{N}$ and $\underline{f} = f_1, \ldots, f_h \in I$ an R-regular sequence. The following statements are equivalent:

(i) $\sqrt{\underline{f}R} = \sqrt{I}$.

(ii) $\mathrm{H}^l_I(R) = 0$ for every $l > h$ and the sequence $\underline{f}$ is quasi-regular on $D(\mathrm{H}^h_I(R))$.

(iii) $\mathrm{H}^l_I(R) = 0$ for every $l > h$ and the sequence $\underline{f}$ is regular on $D(\mathrm{H}^h_I(R))$.

(The case $h = 0$ means

$$\begin{aligned} \sqrt{I} = \sqrt{0} &\iff \mathrm{H}^l_I(R) = 0 \text{ for every } l > 0 \\ &\iff \mathrm{H}^l_I(R) = 0 \text{ for every } l > 0 \text{ and } \Gamma_I(R) \neq 0 \quad). \end{aligned}$$

Proof:

$h = 0$: Clearly the condition $\sqrt{I} = \sqrt{0}$ implies $\mathrm{H}^l_I(R) = 0$ for every $l > 0$. On the other hand, if we have $\mathrm{H}^l_I(R) = 0$ for every $l > 0$, then, by a well-known theorem, one also has $\mathrm{H}^l_I(R/\mathfrak{p}) = 0$ for every prime ideal $\mathfrak{p}$ of R and for every $l > 0$; thus $I \subseteq \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R. But then it is also true that $\Gamma_I(R) = R \neq 0$ holds.

$h \geq 1$: The fact that (i) and (ii) are equivalent follows from theorems 1.1.2 and 1.1.3. Thus we only have to show that (i) implies

$$D(\mathrm{H}^h_I(R))/(f_1, \ldots, f_h)D(\mathrm{H}^h_I(R))) \neq 0 \quad :$$

But, by general Matlis duality theory, the last module is

$$D(\mathrm{Hom}_R(R/(f_1,\ldots,f_h)R, \mathrm{H}^h_I(R))) \quad ;$$

furthermore, every element of $\mathrm{H}^h_I(R)$ is annihilated by a power of I and so it suffices to show $\mathrm{H}^h_I(R) \neq 0$, which is clear, because I is generated up to radical by the regular sequence $f_1,\ldots,f_h$.

1.2 Conjecture (*) on the structure of $\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R)))$

Now, we present an easy property of associated primes of Matlis duals of certain local cohomology modules; this property will naturally lead us to a conjecture on the structure of the set of associated prime ideals of such modules.

1.2.1 Remark

Let $(R,\mathfrak{m})$ be a noetherian local ring, M an R-module, $h \in \mathbb{N}$ and $I \subseteq R$ an ideal such that $\mathrm{H}^l_I(M) = 0$ holds for every $l > h$ and suppose that we have

$$\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^h_I(M))) \quad :$$

This condition clearly implies $\mathrm{Ann}_R(M) \subseteq \mathfrak{p}$ (because $\mathrm{Ann}_R(M)$ is contained in the annihilator of every element of $D(\mathrm{H}^h_I(M))$) and

$$\begin{aligned} 0 &\neq \mathrm{Hom}_R(R/\mathfrak{p}, D(\mathrm{H}^h_I(M))) \\ &= D(\mathrm{H}^h_I(M) \otimes_R (R/\mathfrak{p})) \\ &= D(\mathrm{H}^h_I(M/\mathfrak{p}M)) \quad , \end{aligned}$$

i. e. $\mathrm{H}^h_I(M/\mathfrak{p}M) \neq 0$. In particular $\dim(\mathrm{Supp}_R(M/\mathfrak{p}M)) \geq h$.

As a special case we get the implication

$$\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) \Rightarrow \dim(R/\mathfrak{p}) \geq h$$

for every sequence $x_1,\ldots,x_h \in R$.

Furthermore, as we have seen,

$$\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) | \, \mathrm{H}^h_{(x_1,\ldots,x_h)R}(R/\mathfrak{p}) \neq 0\}$$

holds for every sequence $x_1,\ldots,x_h \in R$.

1.2.2 Conjecture

If $(R,\mathfrak{m})$ is a noetherian local ring, $h > 0$ and $x_1,\ldots,x_h$ are elements of R,

$$(*) \qquad \mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) = \{\mathfrak{p} \in \mathrm{Spec}(R) | \, \mathrm{H}^h_{(x_1,\ldots,x_h)R}(R/\mathfrak{p}) \neq 0\}$$

holds. We denote this conjecture by (*). It is one of the central themes of this work. The next theorem 1.2.3 presents some equivalent characterizations of conjecture (*); one of them is stableness under generalization of the set of associated primes of the Matlis dual of the local cohomology module in question (condition (ii) from theorem 1.2.3). The theorem also shows (condition (iv)) that (*) is actually equivalent to a similar statement, where R is replaced by a finite R-module M, i. e. if (*) holds, a version of (*) also holds for finite R-modules.

1.2.3 Theorem

The following statements are equivalent:

(i) Conjecture (*) holds, i. e. for every noetherian local ring $(R, \mathfrak{m})$, every $h > 0$ and every sequence $x_1, \ldots, x_h$ of elements of R the equality

$$\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) = \{\mathfrak{p} \in \mathrm{Spec}(R) | \, \mathrm{H}^h_{(x_1,\ldots,x_h)R}(R/\mathfrak{p}) \neq 0\}$$

holds.

(ii) For every noetherian local ring $(R, \mathfrak{m})$, every $h > 0$ and every sequence $x_1, \ldots, x_h$ of elements of R the set

$$Y := \mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)}(R)))$$

is stable under generalization, i. e. the implication

$$\mathfrak{p}_0, \mathfrak{p}_1 \in \mathrm{Spec}(R), \mathfrak{p}_0 \subseteq \mathfrak{p}_1, \mathfrak{p}_1 \in Y \Longrightarrow \mathfrak{p}_0 \in Y$$

holds.

(iii) For every noetherian local domain $(R, \mathfrak{m})$, every $h > 0$ and every sequence $x_1, \ldots, x_h$ of elements of R the implication

$$\mathrm{H}^h_{(x_1,\ldots,x_h)}(R) \neq 0 \Longrightarrow \{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R)))$$

holds.

(iv) For every noetherian local ring $(R, \mathfrak{m})$, every finitely generated R-module M, every $h > 0$ and every sequence $x_1, \ldots, x_h$ of elements of R the equality

$$(1) \qquad \mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(M))) = \{\mathfrak{p} \in \mathrm{Supp}_R(M) | \, \mathrm{H}^h_{(x_1,\ldots,x_h)R}(M/\mathfrak{p}M) \neq 0\}$$

holds.

Proof:

First we show that (i) – (iii) are equivalent. (i) $\Longrightarrow$ (ii): In the given situation we have

$$\mathrm{Hom}_R(R/\mathfrak{p}_1, D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) \neq 0 \quad ;$$

this implies

$$\begin{aligned} 0 &\neq \mathrm{Hom}_R(R/\mathfrak{p}_0, D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) \\ &= \mathrm{Hom}_R(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R) \otimes_R (R/\mathfrak{p}_0), \mathrm{E}_R(R/\mathfrak{m})) \\ &= D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R/\mathfrak{p}_0)) \quad . \end{aligned}$$

Thus conjecture (*) implies that $\mathfrak{p}_0$ is associated to $D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))$.

(ii) $\Longrightarrow$ (iii): We assume that $\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R) \neq 0$. This implies $D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R)) \neq 0$ and hence $\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) \neq \emptyset$; now (ii) shows $\{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R)))$.

(iii) $\Longrightarrow$ (i): We have seen above that the inclusion $\subseteq$ holds; we take a prime ideal $\mathfrak{p}$ of R such that $\mathrm{H}^h_{(x_1,\dots,x_h)R}(R/\mathfrak{p}) \neq 0$ and we have to show $\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\dots,x_h)R}(R)))$: We apply (iii) to the domain $R/\mathfrak{p}$ and get an R-linear injection

$$\begin{aligned} R/\mathfrak{p} \to D(\mathrm{H}^h_{(x_1,\dots,x_h)(R/\mathfrak{p})}(R/\mathfrak{p})) \\ &= \mathrm{Hom}_R(\mathrm{H}^h_{(x_1,\dots,x_h)R}(R/\mathfrak{p}), \mathrm{E}_R(R/\mathfrak{m})) \\ &= \mathrm{Hom}_R(\mathrm{H}^h_{(x_1,\dots,x_h)R}(R) \otimes_R R/\mathfrak{p}, \mathrm{E}_R(R/\mathfrak{p})) \\ &= \mathrm{Hom}_R(R/\mathfrak{p}, D(\mathrm{H}^h_{(x_1,\dots,x_h)R}(R))) \\ &\subseteq D(\mathrm{H}^h_{(x_1,\dots,x_h)R}(R)) \quad . \end{aligned}$$

Note that we used $\mathrm{H}^h_{(x_1,\dots,x_h)(R/\mathfrak{p})}(R/\mathfrak{p}) = \mathrm{H}^h_{(x_1,\dots,x_h)R}(R/\mathfrak{p})$ and the fact that $\mathrm{Hom}_R(R/\mathfrak{p}, \mathrm{E}_R(R/\mathfrak{m}))$ is an $R/\mathfrak{p}$-injective hull of $R/\mathfrak{m}$.
Now it is clearly sufficient to show that (i) implies (iv): $\subseteq$: Every element $\mathfrak{p}$ of the left-hand side of identity (1) must contain $\mathrm{Ann}_R(M)$ and hence is an element of $\mathrm{Supp}_R(M)$; furthermore it satisfies

$$\begin{aligned} 0 &\neq \mathrm{Hom}_R(R/\mathfrak{p}, D(\mathrm{H}^h_{(x_1,\dots,x_h)R}(M))) \\ &= \mathrm{Hom}_R(R/\mathfrak{p} \otimes_R \mathrm{H}^h_{(x_1,\dots,x_h)R}(M), \mathrm{E}_R(R/\mathfrak{m})) \\ &= D(\mathrm{H}^h_{(x_1,\dots,x_h)R}(M/\mathfrak{p}M)) \quad . \end{aligned}$$

$\supseteq$: Let $\mathfrak{p}$ be an element of the support of M such that $\mathrm{H}^h_{(x_1,\dots,x_h)R}(M/\mathfrak{p}M)$ is not zero. We set $\overline{R} := R/\mathrm{Ann}_R(M)$, M is an $\overline{R}$-module. $\mathfrak{p} \supseteq \mathrm{Ann}_R(M)$, we set $\overline{\mathfrak{p}} := \mathfrak{p}/\mathrm{Ann}_R(M)$. Clearly our hypothesis implies that $\mathrm{H}^h_{(x_1,\dots,x_h)\overline{R}}(\overline{R}) \neq 0$. We apply (i) to $\overline{R}$ and deduce

$$\overline{\mathfrak{p}} \in \mathrm{Ass}_{\overline{R}}(D(\mathrm{H}^h_{(x_1,\dots,x_h)\overline{R}}(\overline{R}))) \quad .$$

Hence there is an R-linear injection

$$0 \to R/\mathfrak{p} = \overline{R}/\overline{\mathfrak{p}} \to D(\mathrm{H}^h_{(x_1,\dots,x_h)\overline{R}}(\overline{R})) \quad ,$$

which induces an R-linear injection

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M, R/\mathfrak{p}) &\to \mathrm{Hom}_R(M, D(\mathrm{H}^i_{(x_1,\dots,x_h)\overline{R}}(\overline{R}))) \\ &= \mathrm{Hom}_{\overline{R}}(M, D(\mathrm{H}^h_{(x_1,\dots,x_h)\overline{R}}(\overline{R}))) \\ &= D(\mathrm{H}^h_{(x_1,\dots,x_h)\overline{R}}(M)) \\ &= D(\mathrm{H}^h_{(x_1,\dots,x_h)R}(M)) \quad . \end{aligned}$$

Note that for the second equality we have used Hom-Tensor adjointness and for the last equality the facts that M is an $\overline{R}$-module and that $\mathrm{Hom}_R(\overline{R}, \mathrm{E}_R(R/\mathfrak{m}))$ is an $\overline{R}$-injective hull of $R/\mathfrak{m}$; It is sufficient to show $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Hom}_R(M, R/\mathfrak{p}))$; but M is finite and so we have

$$(\mathrm{Hom}_R(M, R/\mathfrak{p}))_{\mathfrak{p}} = \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \neq 0 \quad ,$$

which shows that $\mathfrak{p}R_{\mathfrak{p}}$ is associated to the $R_{\mathfrak{p}}$-module $(\mathrm{Hom}_R(M, R/\mathfrak{p}))_{\mathfrak{p}}$. Thus $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Hom}_R(M, R/\mathfrak{p}))$.

1.2.4 Remark
In [HS1, section 0, conjecture (+)] more was conjectured, namely:

If $(R,\mathfrak{m})$ is a noetherian local ring, $h \geq 1$ and $x_1, \dots, x_h$ is a sequence of elements of R, then all prime ideals $\mathfrak{p}$ maximal in $\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\dots,x_h)R}(R)))$ have the same dimension, namely $\dim(R/\mathfrak{p}) = h$.

This conjecture is false, here is a counterexample:

Let $\mathbf{Q}$ denote the rationals and $R = \mathbf{Q}[[X_1, X_2, X_3, X_4, X_5]]$ a power series algebra over $\mathbf{Q}$ in the variables $X_1, \dots, X_5$. Set $h = 3$ and $x_1 = X_1, x_2 = X_2, x_3 = X_3$. Then

$$p := -X_2X_4^2 + X_3X_4X_5 - X_1X_5^2 + 4X_1X_2 - X_3^2 \in R$$

is a prime element of R; in fact, pR is a maximal element of $\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,x_2,x_3)R}(R)))$, but $\dim(R/fR) = 4 \neq 3$. These statements will be proved in remark 4.3.2 (ii), here we explain where f comes from:

We define a ring

$$S := \mathbf{Q}[[y_1, y_2, y_3, y_4]]$$

and a module-finite $\mathbf{Q}$-algebra homomorphism

$$f : R \to S$$

such that

$$f(X_1) = y_1y_3, f(X_2) = y_2y_4, f(X_3) = y_1y_4 + y_2y_4, f(X_4) = y_1 + y_3, f(X_5) = y_2 + y_4) \quad .$$

As we will see in remark 4.3.2 (ii), pR is the kernel of f; the crucial point here is that the radical of the extension ideal of $(x_1, x_2, x_3)R$ in R_0 is

$$I_0 = (y_1, y_2)R_0 \cap (y_3, y_4)R_0$$

and $\mathrm{H}^3_{I_0}(R_0) \neq 0$, although I_0 has height two (again, see section 4.3 and, in particular, remark 4.3.2 (ii) for details).

1.3 Regular sequences on $D(\mathrm{H}^h_I(R))$ are well-behaved in some sense

Obviously we are dealing here with the notion of regular sequences on modules which are, in general, not finitely generated. Such regular sequences do not have all the good properties of regular sequences on finite modules. However, in our situation, some kind of well-behavior holds, here is the idea (see theorem 1.3.1 below for the precise statement): For finite modules, the following is well-known: If $(R,\mathfrak{m})$ is a noetherian local ring, M a finite R-module and $r_1, \dots, r_h \in R$ an M-regular sequence then $r'_1, \dots, r'_h \in R$ is also an M-regular sequence provided $\sqrt{(r'_1, \dots, r'_h)R} = \sqrt{(r_1, \dots, r_h)R}$ holds (because R is local). In our case, if $(R,\mathfrak{m})$ is a noetherian local ring and $I \subseteq R$ an ideal of R such that $\mathrm{H}^l_I(R) \neq 0 \iff l = h$ holds, it is clear that if an R-regular sequence $r_1, \dots, r_h \in I$ is a $D(\mathrm{H}^h_I(R))$-regular sequence then an R-regular-sequence $r'_1, \dots, r'_h \in I$ is also $D(\mathrm{H}^h_I(R))$-regular if $\sqrt{(r'_1, \dots, r'_h)R} = \sqrt{(r_1, \dots, r_h)R}$ holds (simply because of $\sqrt{(r'_1, \dots, r'_h)R} = \sqrt{I}$ and corollary 1.1.4). But a more sophisticated statement is also true:

1.3.1 Theorem

Let $(R,\mathfrak{m})$ be a noetherian local ring, $h \geq 1$ and $I \subseteq R$ an ideal such that $\mathrm{H}^l_I(R) \neq 0 \iff l = h$ holds. Furthermore, let $1 \leq h' \leq h$ and let $r_1, \dots, r_{h'} \in I$ be an R-regular sequence that is also $D(\mathrm{H}^h_I(R))$-regular.

Furthermore, let $r'_1, \ldots, r'_{h'} \in I$ be such that $\sqrt{(r'_1, \ldots, r'_{h'})R} = \sqrt{(r_1, \ldots, r_{h'})R}$ holds. Then $r'_1, \ldots, r'_{h'}$ is a $D(\mathrm{H}^h_I(R))$-regular sequence. In particular, any permutation of $r_1, \ldots, r_{h'}$ is again a $D(\mathrm{H}^h_I(R))$-regular sequence.
Proof:
R is local, and thus it is clear that $r'_1, \ldots, r'_{h'}$ is an R-regular sequence. By induction on $s \in \{1, \ldots, h'\}$ we show two statements:

$$\mathrm{H}^l_I(R/(r_1, \ldots, r_s)R) \neq 0 \iff l = h - s$$

and

$$D(\mathrm{H}^{h-s}_I(R/(r_1, \ldots, r_s)R)) = D(\mathrm{H}^h_I(R))/(r_1, \ldots, r_s)D(\mathrm{H}^h_I(R)) \quad :$$

$s = 1$: The short exact sequence

$$0 \to R \xrightarrow{r_1} R \to R/r_1R \to 0$$

induces a short exact sequence

$$0 \to \mathrm{H}^{h-1}_I(R/r_1R) \to \mathrm{H}^h_I(R) \xrightarrow{r_1} \mathrm{H}^h_I(R) \to 0$$

and we conclude, therefore, that

$$\mathrm{H}^l_I(R/r_1R) \neq 0 \iff l = h - 1$$

holds. Now, the statement

$$D(\mathrm{H}^{h-1}_I(R/r_1R)) = D(\mathrm{H}^h_I(R))/r_1D(\mathrm{H}^h_I(R))$$

follows from the exactness of D.
$s > 1$: The short exact sequence

$$0 \to R/(r_1, \ldots, r_{s-1})R \xrightarrow{r_s} R/(r_1, \ldots, r_{s-1})R \to R/(r_1, \ldots, r_s)R \to 0$$

induces, by our induction hypothesis, an exact sequence

$$0 \to \mathrm{H}^{h-s}_I(R/(r_1, \ldots, r_s)R) \to \mathrm{H}^{h-(s-1)}_I(R/(r_1, \ldots, r_{s-1})R) \xrightarrow{r_s} \mathrm{H}^{h-(s-1)}_I(R/(r_1, \ldots, r_{s-1})R) \quad .$$

By induction hypothesis,

$$D(\mathrm{H}^{h-(s-1)}_I(R/(r_1, \ldots, r_{s-1})R)) = D(\mathrm{H}^h_I(R))/(r_1, \ldots, r_{s-1})D(\mathrm{H}^h_I(R))$$

and so, by assumption, r_s operates surjectively on $\mathrm{H}^{h-(s-1)}_I(R/(r_1, \ldots, r_{s-1})R)$ and we get

$$\mathrm{H}^l_I(R/(r_1, \ldots, r_s)R) \neq 0 \iff l = r - s$$

and

$$\begin{aligned} D(\mathrm{H}^{h-s}_I(R/(r_1, \ldots, r_s)R)) &= D(\mathrm{H}^{h-(s-1)}_I(R/(r_1, \ldots, r_{s-1})R))/r_sD(\mathrm{H}^{h-(s-1)}_I(R/(r_1, \ldots, r_{s-1})R)) \\ &= D(\mathrm{H}^h_I(R))/(r_1, \ldots, r_s)D(\mathrm{H}^h_I(R)) \quad . \end{aligned}$$

In particular for $s = h'$ we have

$$\mathrm{H}^l_I(R/(r_1, \ldots, r_{h'})R) \neq 0 \iff l = h - h' \quad .$$

Note that, because of

$$\operatorname{depth}(I, R/(r_1, \ldots, r_{h'})R) = \operatorname{depth}(I, R) - h' = \operatorname{depth}(I, R/(r'_1, \ldots, r'_{h'})R)$$

and

$$\operatorname{Supp}_R(R/(r_1, \ldots, r_{h'})R) = \operatorname{Supp}_R(R/(r'_1, \ldots, r'_{h'})R$$

this implies

$$\mathrm{H}^l_I(R/(r'_1, \ldots, r'_{h'})R) \neq 0 \iff l = h - h' \tag{1}$$

(the depth-argument shows vanishing for $l < h - h'$ and the Supp-argument shows that $h - h'$ is the largest number such that $\mathrm{H}^l_I(R/(r'_1, \ldots, r'_{h'})R) \neq 0$). Now, by descending induction on $s \in \{0, \ldots, h' - 1\}$, we will prove the following three statements:

$$r'_{s+1} \text{ operates surjectively on } \mathrm{H}^{h-s}_I(R/(r'_1, \ldots, r'_s)R) \quad ,$$

$$\mathrm{H}^{h-l}_I(R/(r'_1, \ldots, r'_s)R) \neq 0 \iff l = s$$

and

$$D(\mathrm{H}^{h-(s+1)}_I(R/(r'_1, \ldots, r'_{s+1})R)) = D(\mathrm{H}^{h-s}_I(R/(r'_1, \ldots, r'_s)R))/r'_{s+1}D(\mathrm{H}^{h-s}_I(R/(r'_1, \ldots, r'_s)R)) \quad :$$

$s = h' - 1$: We consider the long exact Γ_I-sequence belonging to the short exact sequence

$$0 \to R/(r'_1, \ldots, r'_{h'-1})R \stackrel{r'_{h'}}{\to} R/(r'_1, \ldots, r'_{h'-1})R \to R/(r'_1, \ldots, r'_{h'})R \to 0 \quad :$$

Then, the surjectivity of $r'_{h'}$ on $\mathrm{H}^{h-(h'-1)}_I(R/(r'_1, \ldots, r'_{h'-1})R)$ follows from (1) and the other statements from the fact that for $l \neq h - (h' - 1)$ we have injectivity of $r'_{h'}$ on $\mathrm{H}^l_I(R/(r'_1, \ldots, r'_{h'-1})R)$, hence

$$\mathrm{H}^l_I(R/(r'_1, \ldots, r'_{h'-1})R) = 0$$

as $r'_{h'} \in I$.
$s < h' - 1$: We consider the long exact Γ_I-sequence belonging to the short exact sequence

$$0 \to R/(r'_1, \ldots, r'_s)R \stackrel{r'_{s+1}}{\to} R/(r'_1, \ldots, r'_s)R \to R/(r'_1, \ldots, r'_{s+1})R \to 0 \quad :$$

Then, our induction hypothesis shows that multiplication by r'_{s+1} is surjective on $\mathrm{H}^{h-s}_I(R/(r'_1, \ldots, r'_s))$. Like before, the two other statements follow from the fact that, for $l \neq h - s$, multiplication by r'_{s+1} is injective on $\mathrm{H}^{h-l}_I(R/(r'_1, \ldots, r'_s))$ and so $\mathrm{H}^{h-l}_I(R/(r'_1, \ldots, r'_s))$ is trivial. It is clear that these three statements prove the theorem (in fact, the first and the third statement are sufficient here, the second is used for technical reasons).

1.4 Comparison of two Matlis Duals

For a noetherian local ring $(R, \mathfrak{m})$, the Matlis dual functor clearly depends on R. In this section we will have a local subring R_0 of R. Given any local cohomology module over R, we will take its Matlis dual both with respect to R and with respect to R_0; both are R-modules in a natural way. Among other results, in this section we will see that, under certain assumptions, these two Matlis duals have the same set of associated prime ideals (over R, see 1.4.3 (ii)).

Let $(R,\mathfrak{m})$ be a noetherian local equicharacteristic complete ring with coefficient field k and let $y_1,\dots,y_i$ be a sequence in R such that $R_0 := k[[y_1,\dots,y_i]]$ is regular and of dimension i (this is true, for example, if $\mathrm{H}^i_{(y_1,\dots,y_i)R}(R)\neq 0$ holds, as this local cohomology module agrees with $\mathrm{H}^i_{(y_1,\dots,x_i)R_0}(R_0)\otimes_{R_0} R$; also note that R_0 is, by definition, a subring of R). Let D_R denote the Matlis-dual functor with respect to R and D_{R_0} the one with respect to R_0. By local duality (see e. g. [BS, section 11] for a reference on local duality), we get

$$D_{R_0}(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R)) = \mathrm{Hom}_{R_0}(R\otimes_{R_0}\mathrm{H}^i_{(y_1,\dots,y_i)R_0}(R_0),\mathrm{E}_{R_0}(k)) = \mathrm{Hom}_{R_0}(R,R_0) \quad .$$

$\mathrm{Hom}_{R_0}(R,\mathrm{E}_{R_0}(k))$ is an injective R-module with non-trivial socle; therefore, there exists an injective R-module E' such that

$$\mathrm{Hom}_{R_0}(R,\mathrm{E}_{R_0}(k)) = \mathrm{E}_R(k)\oplus E'$$

holds. We set $E=\Gamma_{(y_1,\dots,y_i)R}(E')$. We have

$$\begin{aligned}
D_{R_0}(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R)) &= \mathrm{Hom}_{R_0}(\mathrm{H}^i_{(y_1,\dots,y_i)R_0}(R_0)\otimes_{R_0} R,\mathrm{E}_{R_0}(k))\\
&= \mathrm{Hom}_{R_0}(\mathrm{H}^i_{(y_1,\dots,y_i)R_0}(R_0),\mathrm{Hom}_{R_0}(R,\mathrm{E}_{R_0}(k)))\\
&= \mathrm{Hom}_{R}(R\otimes_{R_0}\mathrm{H}^i_{(y_1,\dots,y_i)R_0}(R_0),\mathrm{Hom}_{R_0}(R,\mathrm{E}_{R_0}(k)))\\
&= \mathrm{Hom}_{R}(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R),\mathrm{E}_R(k)\oplus E')\\
&= D_R(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R))\oplus\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R),E')\\
&= D_R(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R))\oplus\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R),E)
\end{aligned}$$

and hence

$$(1)\qquad \mathrm{Ass}_R(D_{R_0}(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R))) = \mathrm{Ass}_R(D_R(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R)))\cup\mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R),E)) \quad .$$

It is natural to ask for relations between $D_R(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R))$ and $D_{R_0}(\mathrm{H}^i_{(y_1,\dots,y_i)R}(R))$; we will establish some in the sequel:

For every $\mathfrak{p}\in Z := \{\mathfrak{p}\in\mathrm{Spec}(R)|(y_1,\dots,y_i)R\subseteq\mathfrak{p}\subsetneq\mathfrak{m}\}$ we choose a set $\mu_\mathfrak{p}$ such that

$$E=\bigoplus_{\mathfrak{p}\in Z}\mathrm{E}_R(R/\mathfrak{p})^{(\mu_\mathfrak{p})}$$

holds.

1.4.1 Remark

In the above situation, one has $\mu_\mathfrak{p}\neq\emptyset$ for every $\mathfrak{p}\in Z$.
Proof:
We have to show that $\mathfrak{p}$ is associated to the R-module $\mathrm{Hom}_{R_0}(R/\mathfrak{p},\mathrm{E}_{R_0}(k))$. The latter module is equal to $\mathrm{Hom}_{R_0}(R/\mathfrak{p},k)$, because $\mathfrak{p}$ is annihilated by $y_1,\dots,y_i$ (note that k is the socle of $\mathrm{E}_{R_0}(k)$). Thus we have to prove the following statement: If $(R,\mathfrak{m})$ is a noetherian local equicharacteristic complete domain with coefficient field k, then the zero ideal of R is associated to the R-module $\mathrm{Hom}_k(R,k)$:

Let $x_1, \dots, x_n \in R$ be a system of parameters for R, $n := \dim(R)$. Then $R_0 := k[[x_1, \dots, x_n]]$ is a regular subring of R, over which R is module-finite. One has $\mathrm{Hom}_k(R, k) = \mathrm{Hom}_{R_0}(R, \mathrm{Hom}_k(R_0, k))$ and, therefore, it is sufficient to prove $\{0\} \in \mathrm{Ass}_{R_0}(\mathrm{Hom}_k(R_0, k))$, because in this case, every R_0-injection

$$R_0 \to \mathrm{Hom}_k(R_0, k)$$

induces an R-injection

$$\mathrm{Hom}_{R_0}(R, R_0) \to \mathrm{Hom}_k(R, k)$$

and $\{0\} \in \mathrm{Supp}_R(\mathrm{Hom}_{R_0}(R, R_0))$ holds, because R is finite over R_0. Thus we may assume $R = k[[x_1, \dots, x_n]]$ from now on:
For $i = 1, \dots, n$ we set $R_i := k[[x_1, \dots, x_i]]$. Again we have

$$\mathrm{Hom}_k(R_i, k) = \mathrm{Hom}_{R_{i-1}}(R_i, \mathrm{Hom}_k(R_{i-1}, k))$$

for $i = 2, \dots, n$. Using this and an obvious induction argument, the statement follows from lemma 1.4.2 below.

1.4.2 Lemma
Let k be a field and let $R_0 := k[[X_1, \dots, X_n]]$, $R := k[[X_1, \dots, X_n, X]] = R_0[[X]]$ be power series rings in the variables $X_1, \dots, X_n, X$, respectively. Then

$$\{0\} \in \mathrm{Ass}_R(\mathrm{Hom}_{R_0}(R, R_0)) \quad .$$

Proof:
By $\mathfrak{m}_0$ we denote the maximal ideal of R_0. The canonical short exact sequence

$$0 \to R_0[X] \to R_0[[X]] \to R_0[[X]]/R_0[X] \to 0$$

induces an exact sequence

$$0 \to \mathrm{Hom}_{R_0}(R_0[[X]]/R_0[X], R_0) \to \mathrm{Hom}_{R_0}(R_0[[X]], R_0) \xrightarrow{\alpha} \mathrm{Hom}_{R_0}(R_0[X], R_0) \quad .$$

The map α is the Matlis dual (in the sense that

$$\mathrm{Hom}_{R_0}(\mathrm{H}^n_{\mathfrak{m}_0}(R_0[X]), \mathrm{E}_{R_0}(k)) = \mathrm{Hom}_{R_0}(R_0[X] \otimes_{R_0} \mathrm{H}^n_{\mathfrak{m}_0}(R_0), \mathrm{E}_{R_0}(k)) = \mathrm{Hom}_{R_0}(R_0[X], R_0)$$

and

$$\mathrm{Hom}_{R_0}(\mathrm{H}^n_{\mathfrak{m}_0}(R_0[[X]]), \mathrm{E}_{R_0}(k)) = \mathrm{Hom}_{R_0}(R_0[[X]] \otimes_{R_0} \mathrm{H}^n_{\mathfrak{m}_0}(R_0), \mathrm{E}_{R_0}(k)) = \mathrm{Hom}_{R_0}(R_0[[X]], R_0)$$

hold) of the canonical map

$$\mathrm{H}^n_{\mathfrak{m}_0}(R_0[X]) = \mathrm{H}^n_{\mathfrak{m}_0}(R_0) \otimes_{R_0} R_0[X] \to \mathrm{H}^n_{\mathfrak{m}_0}(R_0) \otimes R_0[[X]] = \mathrm{H}^n_{\mathfrak{m}_0}(R_0[[X]]) \quad ,$$

which is obviously injective. This means that α is surjective. The $R_0[X]$-module $\mathrm{Hom}_{R_0}(R_0[X], R_0)$ can be written as $R_0[[X^{-1}]]$ and in $R_0[[X^{-1}]]$ the element

$$h' := 1 + X^{-1!} + X^{-2!} + \dots$$

has $R_0[X]$-annihilator zero (essentially because the sequence of differences $2!-1!, 3!-2!, \ldots$ becomes arbitrary large). Choose an element $h \in \mathrm{Hom}_{R_0}(R_0[[X]], R_0)$ which is mapped to h' by α Then $\mathrm{ann}_{R_0[X]}(h) = \{0\}$ which implies $\mathrm{ann}_{R_0[[X]]}(h) = \{0\}$, using a flatness argument.

1.4.3 Remarks

Let $(R, \mathfrak{m})$ be a noetherian local ring and $y_1, \ldots, y_i$ a sequence in R and suppose that conjecture (*) holds.

(i) For every fixed prime ideal $\mathfrak{p}$ of R, one has

$$\mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \mathrm{E}_R(R/\mathfrak{p}))) = \{\mathfrak{q} \in \mathrm{Spec}(R) | (\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R/\mathfrak{q}))_\mathfrak{p} \neq 0\} \quad .$$

(ii) For every prime ideal $\mathfrak{p}$ of R, let $\nu_\mathfrak{p}$ be a set. Then

$$\mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \bigoplus_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{E}_R(R/\mathfrak{p})^{(\nu_\mathfrak{p})})) = \bigcup_{\nu_\mathfrak{p}\neq\emptyset} \mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \mathrm{E}_R(R/\mathfrak{p}))) \quad .$$

As a consequence, in the situation of (1) (note that then we had more assumptions: R is complete and equicharacteristic and $\underline{y}$ is such that $\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R) \neq 0$), one has

$$\mathrm{Ass}_R(D_{R_0}(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R))) = \mathrm{Ass}_R(D_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R))) \quad .$$

Proof:

(i) For every prime ideal $\mathfrak{p}$ of R, $\mathrm{E}_R(R/\mathfrak{p}) = \mathrm{E}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ is naturally an $R_\mathfrak{p}$-module. This implies

$$\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \mathrm{E}_R(R/\mathfrak{p})) = \mathrm{Hom}_{R_\mathfrak{p}}(\mathrm{H}^i_{(y_1,\ldots,y_i)R_\mathfrak{p}}(R_\mathfrak{p}), \mathrm{E}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}))$$

and, therefore and because of (*),

$$\begin{aligned}\mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \mathrm{E}_R(R/\mathfrak{p}))) &= \{\mathfrak{P} \cap R | \mathfrak{P} \in \mathrm{Ass}_{R_\mathfrak{p}}(\mathrm{Hom}_{R_\mathfrak{p}}(\mathrm{H}^i_{(y_1,\ldots,y_i)R_\mathfrak{p}}(R_\mathfrak{p}), \mathrm{E}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})))\} \\ &= \{\mathfrak{q} \in \mathrm{Spec}(R) | \, \mathrm{H}^i_{(y_1,\ldots,y_i)R}(R/\mathfrak{q})_\mathfrak{p} \neq 0\} \quad .\end{aligned}$$

(ii) We have natural inclusions

$$\begin{aligned}\bigoplus_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \mathrm{E}_R(R/\mathfrak{p}))^{(\nu_\mathfrak{p})} &\subseteq \mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \bigoplus_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{E}_R(R/\mathfrak{p})^{(\nu_\mathfrak{p})}) \\ &\subseteq \prod_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \mathrm{E}_R(R/\mathfrak{p}))^{\nu_\mathfrak{p}} \quad .\end{aligned}$$

Every annihilator of a non-trivial element of $\prod_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \mathrm{E}_R(R/\mathfrak{p}))^{\nu_\mathfrak{p}}$ is contained in some associated prime ideal of some $\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \mathrm{E}_R(R/\mathfrak{p}))$, where $\nu_\mathfrak{p} \neq \emptyset$. But the set

$$\mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \mathrm{E}_R(R/\mathfrak{p})))$$

is stable under generalization because of the conjecture (*). Therefore, we get

$$\begin{aligned}\mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \bigoplus_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{E}_R(R/\mathfrak{p})^{(\nu_\mathfrak{p})})) &= \mathrm{Ass}_R(\bigoplus_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \mathrm{E}_R(R/\mathfrak{p}))^{(\nu_\mathfrak{p})}) \\ &= \bigcup_{\mathfrak{p}\in Z} \mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R), \mathrm{E}_R(R/\mathfrak{p}))) \\ &\subseteq \mathrm{Ass}_R(D_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R))) \quad .\end{aligned}$$

In particular, in the situation of (1), we have

$$\mathrm{Ass}_R(D_{R_0}(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R))) = \mathrm{Ass}_R(D_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R))) \quad .$$

2 Associated primes – a constructive approach

In this section we will prove results on the set

$$\mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))) \quad ,$$

where $\underline{x} = x_1, \ldots, x_i$ is a sequence in a noetherian local ring R. The proofs are based on the fact that, over the formal power series ring $R = k[[X_1, \ldots, X_n]]$ (k a field), the R-module

$$E = k[X_1^{-1}, \ldots, X_n^{-1}]$$

is an R-injective hull of k. The methods in this sections are constructive to some extent, in fact, we construct certain elements in $k[X_1^{-1}, \ldots, X_n^{-1}]$. For the proofs, we will have to distinguish between the equicharacteristic and the mixed-characteristic case. One major result in this section is (theorem 2.4, see also theorem 2.5 for the case of mixed characteristic):
If $\underline{x} = x_1, \ldots x_i$ is a sequence in a noetherian local equicharacteristic ring $(R, \mathfrak{m})$ and $\underline{x}$ is part of a system of parameters of $R/\mathfrak{p}$ for some fixed prime ideal $\mathfrak{p}$ of R, then one has

$$\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))) \quad .$$

We will also see that, in general, not all associated primed of $D := D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))$ are obtained in this way (remark 2.7 (ii)). As a corollary, we are able to completely compute the set $\mathrm{Ass}_R(D)$ in the case $i = 1$ (corollary 2.6):

$$\mathrm{Ass}_R(D(\mathrm{H}^1_{xR}(R))) = \mathrm{Spec}(R) \setminus \mathcal{V}(x)$$

(note that $\mathcal{V}(x) = \{\mathfrak{p} \in \mathrm{Spec}(R) | x \in \mathfrak{p}\}$). In particular, this set is infinite (in general). The sections ends with remarks on the questions of stableness under generalization of the subsets

$$Z_1 := \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \mathrm{H}^i_{(x_1,\ldots,x_i)R}(R/\mathfrak{p}) \neq 0\}$$

and

$$Z_2 := \{\mathfrak{p} \in \mathrm{Spec}(R) | x_1, \ldots, x_i \text{ is part of a system of parameters for } R/\mathfrak{p}\}$$

of $\mathrm{Spec}(R)$. Note that we have

$$Z_2 \subseteq \mathrm{Ass}_R(D) \subseteq Z_1$$

by theorems 2.4, 2.5 and remark 1.1.2.

We start with a special case of the result mentioned above:

2.1 Lemma

Let k be a field, $n \geq 1$, $R = k[[X_1, \ldots, X_n]]$ and $i \in \{1, \ldots, n\}$. We set $I := (X_1, \ldots, X_i)R$ and $\mathfrak{m} := (X_1, \ldots, X_n)R$. Then

$$\{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^i_I(R)))$$

holds.
Proof:
1. Case: $i = n$:

Here $\mathrm{H}^i_I(R) = \mathrm{E}_R(R/\mathfrak{m})$ und also $D(\mathrm{H}^i_I(R)) = R$ and the statement follows.
2. Case: $i < n$: We have

$$\mathrm{H}^i_I(R) = \varinjlim_{l \in \mathbf{N}\setminus\{0\}} (R/(X_1^l, \ldots, X_i^l)R) \quad ,$$

the transition maps being induced by $R \to R$, $r \mapsto (X_1 \cdot \ldots \cdot X_i) \cdot r$. So

$$D(\mathrm{H}^i_I(R)) = \varprojlim_{l \in \mathbf{N}\setminus\{0\}} (D(R/(X_1^l, \ldots, X_i^l)R)) \quad ;$$

here

$$D(R/(X_1^l, \ldots, X_i^l)R) = \mathrm{Hom}_R(R/(x_1^l, \ldots, x_i^l), D(R)) = \mathrm{E}_{R/(X_1^l,\ldots,X_i^l)R}(R/\mathfrak{m})(\subseteq \mathrm{E}_R(R/\mathfrak{m})) \quad ,$$

the transition maps being induced by $\mathrm{E}_R(R/\mathfrak{m}) \to \mathrm{E}_R(R/\mathfrak{m})$, $e \mapsto (X_1 \cdot \ldots \cdot X_i) \cdot e$ and we have $\mathrm{E}_R(R/\mathfrak{m}) = k[X_1^{-1}, \ldots, X_n^{-1}]$ (by definition, the last module is the k-vector space with basis $(X_1^{i_1} \cdot \ldots \cdot X_n^{i_n})_{i_1,\ldots,i_n \leq 0}$ and with an obvious R-module structure on it). We define

$$\begin{aligned}\alpha :=& (1, X_1^{-1} \cdot \ldots \cdot X_i^{-1} + X_{i+1}^{-1!} \cdot \ldots \cdot X_n^{-1!}, \ldots, X_1^{-m} \cdot \ldots \cdot X_i^{-m} + \\ &+ (X_{i+1}^{-1!} \cdot \ldots \cdot X_n^{-1!}) \cdot (X_1^{-(m-1)} \cdot \ldots \cdot X_i^{-(m-1)}) + \ldots + \\ &+ (X_{i+1}^{-(m-1)!} \cdot \ldots \cdot X_n^{-(m-1)!}) \cdot (X_1^{-1} \cdot \ldots \cdot X_i^{-1}) + X_{i+1}^{-m!} \cdot \ldots \cdot X_n^{-m!}, \ldots) \in D(\mathrm{H}^i_I(R)) \quad .\end{aligned}$$

Here we consider the projective limit as a subset of a direct product. We state $\mathrm{ann}_R(\alpha) = \{0\}$: Assume there is an $f \in \mathrm{ann}_R(\alpha) \setminus \{0\}$. We choose $(a_1, \ldots, a_n) \in \mathrm{Supp}(f)$ such that $(a_1, \ldots, a_i)$ is minimal (using the ordering

$$(c_1, \ldots c_i) \leq (c_1', \ldots, c_i') :\iff c_1 \leq c_1' \wedge \ldots \wedge c_i \leq c_i' \quad)$$

in

$$\{(a_1', \ldots, a_i') | \exists a_{i+1}', \ldots, a_n' : (a_1', \ldots, a_n') \in \mathrm{Supp}(f)\} \quad .$$

We may assume $a_1 = \max\{a_1, \ldots, a_i\}$. We replace f by $X_2^{a_1-a_2} \cdot \ldots \cdot X_i^{a_1-a_i} \cdot f$; this means $a_1 = \ldots = a_i =: a$. Choose $h_1, \ldots, h_i \in R$ and $g \in k[[X_{i+1}, \ldots, X_n]] \setminus \{0\}$ such that

$$f = X_1^{a+1} h_1 + \ldots + X_i^{a+1} h_i + (X_1^a \cdot \ldots \cdot X_i^a) \cdot g$$

$f \cdot \alpha = 0$ means: For every m we have

$$\begin{aligned} 0 &= [X_1^{a+1} h_1 + \ldots + X_i^{a+1} h_i + (X_1^a \cdot \ldots \cdot X_i^a) \cdot g] \cdot (X_1^{-m} \cdot \ldots \cdot X_i^{-m} + \ldots + X_{i+1}^{-m!} \cdot \ldots \cdot X_n^{-m!}) \\ &= (X_1^{a+1} h_1 + \ldots + X_i^{a+1} h_i) \cdot [X_1^{-m} \cdot \ldots \cdot X_i^{-m} + \ldots + (X_{i+1}^{-(m-a-1)!} \cdot \ldots \cdot X_n^{-(m-a-1)!}) \cdot \\ &\cdot (X_1^{-(a+1)} \cdot \ldots \cdot X_i^{-(a+1)})] + g \cdot (X_1^{-(m-a)} \cdot \ldots \cdot X_i^{-(m-a)} + \ldots + X_{i+1}^{-(m-a)!} \cdot \ldots \cdot X_n^{-(m-a)!}) \quad . \end{aligned}$$

Choose $(b_{i+1}, \ldots, b_n)$ minimal in $\mathrm{Supp}(g)$; then for all $m >> 0$ the following statements must hold:

$$\begin{gathered}(m-a)! - b_{i+1} \leq (m-a-1)! \\ \vdots \\ (m-a)! - b_n \leq (m-a-1)!\end{gathered}$$

For $m >> 0$ this leads to a contradiction, the assumption is wrong and the lemma is proven.

2.2 Lemma

Let p be a prime number, C a complete p-ring, $n \geq 1$, $R = C[[X_1, \ldots, X_n]]$ and $i \in \{1, \ldots, n\}$. We set $I := (X_1, \ldots, X_i)R$ and $\mathfrak{m} := (p, X_1, \ldots, X_n)R$. Then

$$\{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^i_I(R)))$$

holds.

Proof:

We have

$$\mathrm{H}^i_I(R) = \varinjlim_{l \in \mathbf{N} \setminus \{0\}} (R/(X_1^l, \ldots, X_i^l)R) \quad ,$$

the transition maps being induced by $R \to R$, $r \mapsto (X_1 \cdot \ldots \cdot X_i) \cdot r$. We deduce

$$D(\mathrm{H}^i_I(R)) = \varprojlim_{l \in \mathbf{N} \setminus \{0\}} (D(R/(X_1^l, \ldots, X_i^l)R)) \quad ;$$

we recall

$$D(R/(X_1, \ldots, X_i^l)R) = \mathrm{E}_{R/(X_1^l, \ldots X_i^l)R}(R/\mathfrak{m})(\subseteq \mathrm{E}_R(R/\mathfrak{m})) \quad ,$$

the transition maps being induced by $\mathrm{E}_R(R/\mathfrak{m}) \to \mathrm{E}_R(R/\mathfrak{m})$, $e \mapsto (X_1 \cdot \ldots \cdot X_i) \cdot e$. Furthermore

$$\mathrm{E}_R(R/\mathfrak{m}) = (C_p/C)[X_1^{-1}, \ldots, X_n^{-1}]$$

holds (because of

$$\begin{aligned} \mathrm{E}_R(R/\mathfrak{m}) &= \mathrm{H}^{n+1}_{(p, X_1, \ldots, X_n)R}(R) \\ &= \mathrm{H}^1_{pR}(R) \otimes_R \ldots \otimes_R \mathrm{H}^1_{X_n R}(R) \\ &= (C_p/C) \otimes_C ((R_{X_1}/R) \otimes_R \ldots \otimes_R (R_{X_n}/R)) \quad . \end{aligned}$$

We define

$$\begin{aligned} \alpha := &(p^{-1}, p^{-1}X_1^{-1} \cdot \ldots \cdot X_i^{-1} + p^{-1!}X_{i+1}^{-1!} \cdot \ldots \cdot X_n^{-1!}, \ldots, p^{-1}X_1^{-m} \cdot \ldots \cdot X_i^{-m} + \\ &+ (p^{-1!}X_{i+1}^{-1!} \cdot \ldots \cdot X_n^{-1!}) \cdot (X_1^{-(m-1)} \cdot \ldots \cdot X_i^{-(m-1)}) + \ldots + \\ &+ (p^{-(m-1)!}X_{i+1}^{-(m-1)!} \cdot \ldots \cdot X_n^{-(m-1)!}) \cdot (X_1^{-1} \cdot \ldots \cdot X_i^{-1}) + p^{-m!}X_{i+1}^{-m!} \cdot \ldots \cdot X_n^{-m!}, \ldots) \in D(\mathrm{H}^i_I(R)) \end{aligned}$$

and, similar to the proof of lemma 2.1, we show that $\mathrm{ann}_R(\alpha) = 0$. Assume to the contrary there is an $f \in \mathrm{ann}_R(\alpha) \setminus \{0\}$. Choose $(a_1, \ldots, a_i)$ minimal in

$$\{(a'_1, \ldots, a'_i) | \text{there exists } a'_{i+1}, \ldots a'_n \text{ such that } (a'_1, a'_n) \in \mathrm{Supp}(f)\} \quad .$$

Like before we may assume $a_1 = \ldots = a_i =: a$. Choose $h_1, \ldots, h_i \in R$ and $g \in C[[X_{i+1}, \ldots, X_n]] \setminus \{0\}$ such that

$$f = X_1^{a+1} \cdot h_1 + \ldots + X_i^{a+1} \cdot h_i + X_1^a \cdot \ldots \cdot X_i^a \cdot g$$

$\alpha \cdot f = 0$ implies, for all $m \in \mathbf{N} \setminus \{\mathbf{0}\}$,

$$0 = (X_1^{a+1}h_1 + \ldots + X_i^{a+1}h_i + X_1^a \cdot \ldots \cdot X_i^a \cdot g) \cdot (p^{-1}X_1^{-m} \cdot \ldots \cdot X_i^{-m} + \ldots + p^{-m!}X_{i+1}^{-m!} \cdot \ldots \cdot X_n^{-m!}) =$$

$$= (X_1^{a+1}h_1 + \ldots + X_i^{a+1}h_i) \cdot [p^{-1}X_1^{-m} \cdot \ldots \cdot X_i^{-m} + \ldots + (p^{-(m-a-1)!} \cdot X_{i+1}^{-(m-a-1)!} \cdot \ldots \cdot X_n^{-(m-a-1)!}) \cdot$$

$$\cdot(X_1^{-(a+1)}\cdot\ldots\cdot X_i^{-(a+1)})] + g\cdot(p^{-1}X_1^{-(m-a)}\cdot\ldots\cdot X_i^{-(m-a)} + \ldots + p^{-(m-a)!}X_{i+1}^{-(m-a)!}\cdot\ldots\cdot X_n^{-(m-a)!}) \quad .$$

Now, let $(b_{i+1},\ldots,b_n)$ be minimal in $\mathrm{Supp}(g)$ and $c\in C$ be the coefficient of g in front of $X_{i+1}^{b_{i+1}}\cdot\ldots\cdot X_n^{b_n}$. In C_p/C we have $c\cdot p^{-(m-a)!}\neq 0$ for all $m>>0$. So, like before, we must have

$$(m-a)! - b_{i+1} \leq (m-a-1)!$$

$$(m-a)! - b_n \leq (m-a-1)!$$

for all $m>>0$, which leads to a contradiction again.

2.3 Lemma

Let p be a prime number, C a complete p-ring, $n\in\mathbf{N}$, $R=C[[X_1,\ldots,X_n]]$, $i\in\{0,\ldots,n\}$, $I:=(p,X_1,\ldots,X_i)R$ and $\mathfrak{m}:=(p,X_1,\ldots,X_n)R$. Then

$$\{0\}\in\mathrm{Ass}_R(D(\mathrm{H}_I^{i+1}(R)))$$

holds.

Proof:

1. Case: $i=n$: In this case we have $\mathrm{H}_I^{i+1}(R)=\mathrm{E}_R(R/\mathfrak{m})$ and hence $D(\mathrm{H}_I^{i+1}(R))=R$.
2. Case: $i<n$: Similar to the situation in the proof of lemma we have

$$D(\mathrm{H}_I^{i+1}(R)) = \varprojlim(\mathrm{E}_{R/(p,X_1,\ldots,X_i)R}(R/\mathfrak{m}) \overset{p\cdot X_1\cdot\ldots\cdot X_i}{\longleftarrow} \mathrm{E}_{R/(p^2,X_1^2,\ldots,X_i^2)R}(R/\mathfrak{m}) \overset{p\cdot X_1\cdot\ldots\cdot X_i}{\longleftarrow} \ldots)$$

$$\mathrm{E}_R(R/\mathfrak{m}) = (C_p/C)[X_1^{-1},\ldots,X_n^{-1}]$$

and we define

$$\begin{aligned}\alpha :=&(p^{-1},p^{-2}X_1^{-1}\cdot\ldots\cdot X_i^{-1}+p^{-2}X_{i+1}^{-1!}\cdot\ldots\cdot X_n^{-1!},\ldots,p^{-(m+1)}X_1^{-m}\cdot\ldots\cdot X_i^{-m}+\\ &+p^{-(m+1)}X_{i+1}^{-1!}\cdot\ldots\cdot X_n^{-1!}\cdot X_1^{-(m+1)}\cdot\ldots\cdot X_i^{-(m-1)}+\ldots+\\ &+p^{-(m+1)}X_{i+1}^{-(m-1)!}\cdot\ldots\cdot X_n^{-(m-1)!}\cdot X_1^{-1}\cdot\ldots\cdot X_i^{-1}+p^{-(m+1)}X_{i+1}^{-m!}\cdot\ldots\cdot X_n^{-m!},\ldots) \quad .\end{aligned}$$

Again we state $\mathrm{ann}_R(\alpha)$ and assume, to the contrary, that there exists an $f\in\mathrm{ann}_R(\alpha)\setminus\{0\}$, choose $(a_1,\ldots,a_i)$ minimal in

$$\{(a_1',\ldots,a_i')|\text{There exist } a_{i+1}',\ldots,a_n' \text{ such that } (a_1',\ldots,a_n')\in\mathrm{Supp}(f)\}$$

may assume $a_1=\ldots=a_i=:a$ and choose $h_1,\ldots,h_i\in R$, $g\in C[[X_{i+1},\ldots,X_n]]$ such that

$$f = X_1^{a+1}h_1+\ldots+X_i^{a+1}h_i+(X_1^a\cdot\ldots\cdot a_i^a)\cdot g \quad .$$

This means, for all $m\in\mathbf{N}$,

$$0=(X_1^{a+1}h_1+\ldots+X_i^{a+1}h_i)[p^{-(m+1)}X_1^{-m}\cdot\ldots\cdot X_i^{-m}+\ldots+(p^{-(m+1)}X_{i+1}^{-(m-a-1)!}\cdot\ldots\cdot X_n^{-(m-a-1)!})\cdot$$

$$\cdot(X_1^{-(a+1)}\cdot\ldots\cdot X_i^{-(a+1)})]+g\cdot(p^{-(m+1)}X_1^{-(m-a)}\cdot\ldots\cdot X_i^{-(m-a)}+\ldots+p^{-(m+1)}X_{i+1}^{-(m-a)!}\cdot\ldots\cdot X_n^{-(m-a)!}) \quad .$$

Choose $(b_{i+1},\ldots,b_n)$ minimal in $\mathrm{Supp}(g)$ and let $c\in C$ be the coefficient of g in front of $X_{i+1}^{b_{i+1}}\cdot\ldots\cdot X_n^{b_n}$. In C_p/C we have $g\cdot p^{-(m+1)}\neq 0$ for all $m>>0$, and so we must have for all $m>>0$

$$(m-a)! - b_{i+1} \leq (m-a-1)!$$

$$(m-a)! - b_n \leq (m-a-1)!$$

which leads to a contradiction, proving the lemma.

Now we are ready to prove that certain prime ideals are associated to $D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))$ in a more general situation (R does not have to be regular). This is done essentially by using various base-change arguments and lemmas 2.1 – 2.3:

2.4 Theorem
Let $(R,\mathfrak{m})$ be a noetherian local ring, $i \geq 1$ and $x_1,\ldots,x_i$ a sequence on R. Then

$$\mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \mathrm{H}^i_{(x_1,\ldots,x_i)R}(R/\mathfrak{p}) \neq 0\}$$

holds. If R is equicharacteristic,

$$\{\mathfrak{p} \in \mathrm{Spec}(R) | x_1,\ldots,x_i \text{ is part of a system of parameters for } R/\mathfrak{p}\} \subseteq \mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)))$$

holds.
Proof:
The first inclusion was shown in remark 1.2.1. For the second inclusion let $\mathfrak{p} \in \mathrm{Spec}(R)$ and $x_{i+1},\ldots,x_n \in R$ such that $x_1,\ldots,x_n$ (more precisely: their images in $R/\mathfrak{p}$) form a system of parameters for $R/\mathfrak{p}$; then $n = \dim(R/\mathfrak{p})$. $x_1,\ldots,x_n$ also form a system of parameters in $\hat{R}/\mathfrak{p}\hat{R}$. Choose $\mathfrak{q} \in \mathrm{Spec}(\hat{R})$ with $\dim(\hat{R}/\mathfrak{q}) = \dim(R/\mathfrak{p})$. This implies $\mathfrak{q} \in \mathrm{Min}(\hat{R})$ and $\mathfrak{q} \cap R = \mathfrak{p}$. Because of $\dim(\hat{R}/\mathfrak{q}) = \dim(R/\mathfrak{p})$ the elements $x_1,\ldots,x_n$ form a system of parameters of $\hat{R}/\mathfrak{q}$. It is sufficient to show $\mathfrak{q} \in \mathrm{Ass}_{\hat{R}}(D(\mathrm{H}^i_{(x_1,\ldots,x_i)\hat{R}}(\hat{R})))$. Namely, as

$$\begin{aligned}
D(\mathrm{H}^i_{(x_1,\ldots,x_i)\hat{R}}(\hat{R})) &= \mathrm{Hom}_{\hat{R}}(\mathrm{H}^i_{(x_1,\ldots,x_i)\hat{R}}(\hat{R}), \mathrm{E}_{\hat{R}}(\hat{R}/\mathfrak{m}\hat{R})) \\
&= \mathrm{Hom}_{\hat{R}}(\mathrm{H}^i_{(x_1,\ldots,x_i)\hat{R}}(\hat{R}), \mathrm{E}_R(R/\mathfrak{m})) \\
&= \mathrm{Hom}_{\hat{R}}(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R) \otimes_R \hat{R}, \mathrm{E}_R(R/\mathfrak{m})) \\
&= \mathrm{Hom}_R(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R), \mathrm{E}_R(R/\mathfrak{m})) \\
&= D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)) \quad ,
\end{aligned}$$

every monomorphism $\hat{R}/\mathfrak{q} \to D(\mathrm{H}^i_{(x_1,\ldots,x_i)\hat{R}}(\hat{R}))$ induces a monomorphism

$$R/\mathfrak{p} \stackrel{\text{kan.}}{\to} \hat{R}/\mathfrak{q} \to D(\mathrm{H}^i_{(x_1,\ldots,x_i)\hat{R}}(\hat{R})) = D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)) \quad .$$

This means we may assume that R is complete.
We have to show that the zero ideal of $R/\mathfrak{p}$ is associated to

$$\mathrm{Hom}_R(R/\mathfrak{p}, D(\mathrm{H}^i_{x_1,\ldots,x_i)R}(R)) = D(\mathrm{H}^i_{(x_1,\ldots,x_i)R/\mathfrak{p}}(R/\mathfrak{p}))$$

(this equality was shown in the proof of the first inclusion). Replacing R by $R/\mathfrak{p}$ we may assume that R is a domain and $\mathfrak{p}$ is the zero ideal in R. Let $k \subseteq R$ denote a coefficient field.

$$R_0 := k[[x_1,\ldots,x_n]] \subseteq R$$

is an n-dimensional regular local subring of R, over which R is module-finite. Let $\mathfrak{m}_0$ denote the maximal ideal of R_0. The R-Modul $\operatorname{Hom}_{R_0}(R, \mathrm{E}_{R_0}(R_0/\mathfrak{m}_0))$ is isomorphic to $\mathrm{E}_R(R/\mathfrak{m})$. We have

$$\begin{aligned} D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)) &= \operatorname{Hom}_R(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R), \mathrm{E}_R(R/\mathfrak{m})) \\ &= \operatorname{Hom}_R(\mathrm{H}^i_{(x_1,\ldots,x_i)R_0}(R_0) \otimes_{R_0} R, \mathrm{E}_R(R/\mathfrak{m})) \\ &= \operatorname{Hom}_{R_0}(\mathrm{H}^i_{(x_1,\ldots,x_i)R_0}(R_0), \operatorname{Hom}_{R_0}(R, \mathrm{E}_{R_0}(R_0/\mathfrak{m}_0))) \\ &= \operatorname{Hom}_{R_0}(R, \operatorname{Hom}_{R_0}(\mathrm{H}^i_{(x_1,\ldots,x_i)R_0}(R_0), \mathrm{E}_{R_0}(R_0/\mathfrak{m}_0))) \\ &= \operatorname{Hom}_{R_0}(R, D(\mathrm{H}^i_{(x_1,\ldots,x_i)R_0}(R_0))) \quad . \end{aligned}$$

By lemma 2.1 there exists a monomorphism $R_0 \to D(\mathrm{H}^i_{(x_1,\ldots,x_i)R_0}(R_0))$; so we get a monomorphism

$$\operatorname{Hom}_{R_0}(R, R_0) \to \operatorname{Hom}_{R_0}(R, D(\mathrm{H}^i_{(x_1,\ldots,x_i)R_0}(R_0))) = D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)) \quad .$$

R is a domain and module-finite over R_0, and thus $\{0\} \in \operatorname{Supp}_R(\operatorname{Hom}_{R_0}(R, R_0))$; the statement now follows.

Again, there are versions for the case of mixed characteristic:

2.5 Theorem
Let $(R, \mathfrak{m})$ be a noetherian local ring of mixed characteristic, $p = \operatorname{char}(R/\mathfrak{m})$, $i \geq 0$ and $x_1, \ldots, x_i \in R$. Then

$$\{\mathfrak{p} \in \operatorname{Spec}(R) | p, x_1, \ldots, x_i \text{ is part of a system of parameters for } R/\mathfrak{p}\} \subseteq \operatorname{Ass}_R(D(\mathrm{H}^{i+1}_{(p,x_1,\ldots,x_i)R}(R))) \quad .$$

In case $i \geq 1$, we have in addition

$$\{\mathfrak{p} \in \operatorname{Spec}(R) | p, x_1, \ldots, x_i \text{ is part of a system of parameters for } R/\mathfrak{p}\} \subseteq \operatorname{Ass}_R(D(\mathrm{H}^{i}_{(x_1,\ldots,x_i)R}(R))) \quad .$$

Theorem 2.5 is proved in a similar way like Theorem 2.4, using lemmas 2.2 and 2.3 instead of lemma 2.1.

In the case $i = 1$ the results proven so far are sufficient to completely compute the set of associated primes:

2.6 Corollary
Let $(R, \mathfrak{m})$ be a noetherian local equicharacteristic ring and $x \in R$. Then

$$\operatorname{Ass}_R(D(\mathrm{H}^1_{xR}(R))) = \operatorname{Spec}(R) \setminus \mathfrak{V}(x)$$

holds. In particular, this set is infinite in general.

2.7 Remarks
(i) If one has $\operatorname{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))) = \emptyset$ in the situation of the theorem, it follows that $\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R) = 0$ and also $\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R/\mathfrak{p}) = 0$ for every $\mathfrak{p} \in \operatorname{Spec}(R)$ (by a well-known theorem), i. e. in this case conjecture (*) holds.

(ii) The second inclusion of theorem 2.4 is not an equality in general: For a counterexample let k be a field, $R = k[[y_1, y_2, y_3, y_4]]$ and define $x_1 = y_1y_3$, $x_2 = y_2y_4$, $x_3 = y_1y_4 + y_2y_3$. x_1, x_2, x_3 is not part of a system of parameters for R, but we have

$$\sqrt{(x_1, x_2, x_3)R} = (y_1, y_2)R \cap (y_3, y_4)R$$

and so a Mayer-Vietoris sequence argument (see, e. g. [BS, 3.2.3] for a reference on the Mayer-Vietoris sequence) shows

$$\mathrm{E}_R(k) = \mathrm{H}^3_{(y_1,y_2)R\cap(y_3,y_4)R}(R) = \mathrm{H}^3_{(x_1,x_2,x_3)R}(R)$$

and so $D(\mathrm{H}^3_{(x_1,x_2,x_3)R}(R)) = R$. Thus $\{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^3_{(x_1,x_2,x_3)R}(R)))$.
(iii) In the situation of theorem 2.4, set

$$Z_1 := \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \mathrm{H}^i_{(x_1,\ldots,x_i)R}(R/\mathfrak{p}) \neq 0\}$$

and

$$Z_2 := \{\mathfrak{p} \in \mathrm{Spec}(R) | x_1,\ldots,x_i \text{ is part of a system of parameters for } R/\mathfrak{p}\} \quad .$$

Then Z_1 is stable under generalization (this follows e. g. from the following well-known fact: If I is an ideal of a noetherian domain R such that $0 = \mathrm{H}^l_I(R) = \mathrm{H}^{l+1}_I(R) = \ldots$ holds for some fixed $l \in \mathbb{N}$, then $0 = \mathrm{H}^l_I(M) = \mathrm{H}^{l+1}_I(M) = \ldots$ holds for every R-module M).
But note that, in general, $Z_2 \subseteq \mathrm{Spec}(R)$ is not stable under generalization, even not if R is regular; namely, for an example where Z_2 is not stable under generalization, let $R = k[[x_1,x_2,x_3,x_4]]$ be a formal power series algebra in four variables over a field k, set

$$\mathfrak{p}_0 = (x_1x_4 + x_2x_3)R \text{ and } \mathfrak{p} = (x_3,x_4)R \quad .$$

Then x_1,x_2 is a system of parameters for $R/\mathfrak{p}$, but is not a part for $R/\mathfrak{p}_0$ (because $x_1x_4+x_2x_3$ is contained in the ideal (x_1,x_2)), i. e. we have

$$\mathfrak{p}_0 \subseteq \mathfrak{p}, \mathfrak{p} \in Z_2, \mathfrak{p}_0 \notin Z_2 \quad .$$

Assume now that R is regular; then, at least, the following special form of stableness (of Z_2) under generalization holds: Let $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $x_1,\ldots,x_i$ is part of a system of parameters for $R/\mathfrak{p}$. Then $x_1,\ldots,x_i$ is part of a system of parameters for R, i. e. one has the implication

$$Z_2 \neq \emptyset \Longrightarrow \{0\} \in Z_2 \quad .$$

This follows from the so-called height-formula which holds for regular local rings and which says (we apply it to the ideal $(x_1,\ldots,x_i)R + \mathfrak{p} \subseteq R$):

$$\mathrm{height}((x_1,\ldots,x_i)R + \mathfrak{p}) \leq \mathrm{height}((x_1,\ldots,x_i)R) + \mathrm{height}(\mathfrak{p}) \leq i + \mathrm{height}(\mathfrak{p}) \quad .$$

But, because of our assumption $\mathfrak{p} \in Z_2$, we must have

$$\mathrm{height}((x_1,\ldots,x_i)R + \mathfrak{p}) = i + \mathrm{height}(\mathfrak{p})$$

and, therefore, $\mathrm{height}((x_1,\ldots,x_i)R) = i$.

3 Associated primes – the characteristic-free approach

In this section we investigate associated prime ideals of Matlis duals $D(\mathrm{H}^i_I(M))$ of local cohomology modules (R is local, of course); there are two subsections: In the first one, we prove characteristic-free versions of some results on the set of associated primes of such a module; out methods here are different to the ones used in section 2. Some results of this section can be found in [HS1]. In the second part of this section, we concentrate on the case $M = R$, $i = \dim(R) - 1$, theorems 3.2.6 and 3.2.7 (where we actually compute the set of associated primes of $D(\mathrm{H}^{\dim(R)-1}_I(R))$) contain the main results of this second subsection.

3.1 Characteristic-free versions of some results

The following lemma is crucial for this subsection:

3.1.1 Lemma

Let R be a ring, $x, y \in R$ and U an R-submodule of R_x such that $\mathrm{im}\,\iota_x \subseteq U$, where $\iota_x : R \to R_x$ is the canonical map. Let $S := \mathrm{im}\,\iota_y \subseteq R_y$. There exists an R-epimorphism

$$R_x/U \to R_{xy}/(S_x + U_y).$$

Proof:

Let $V := S_x + U_y \subseteq R_{xy}$ and let $(b_1, b_2, \ldots) \in R^{\mathbb{N}^+}$ be an infinite sequence. For $i \in \mathbb{N}$ we set

$$\rho_i := \sum_{j=1}^{i} \frac{b_j}{x^{i-j+1}y^j} + V \in R_{xy}/V \quad (i \in \mathbb{N}).$$

We calculate

$$\begin{aligned} x\rho_{i+1} - \rho_i &= \Big(\sum_{j=1}^{i+1} \frac{xb_j}{x^{i-j+2}y^j} + V\Big) - \Big(\sum_{j=1}^{i} \frac{b_j}{x^{i-j+1}y^j} + V\Big) \\ &= \frac{b_{i+1}}{y^{i+1}} + V \\ &= 0, \end{aligned}$$

because

$$\frac{b_{i+1}}{y^{i+1}} \in (\mathrm{im}\,\iota_x)_y \subseteq U_y \subseteq V \quad .$$

Thus we have $x\rho_{i+1} = \rho_i$ for all $i \in \mathbb{N}$ and so we get a map $\varphi : R_x \to R_{xy}/V$ given by

$$\frac{r}{x^i} \mapsto r\rho_i \quad (r \in R, i \in \mathbb{N}).$$

It is easy to see that φ is R-linear. Let $u \in U$ be arbitrary. There are $r \in R$ and $i \in \mathbb{N}$ such that $u = \frac{r}{x^i}$. We have

$$\varphi(u) = r\rho_i = \sum_{j=1}^{i} \frac{rb_j}{x^{i-j+1}} + V = u\sum_{j=1}^{i} \frac{x^{j-1}b_j}{y^j} + V = 0,$$

because

$$u\sum_{j=1}^{i} \frac{x^{j-1}b_j}{y^j} \in U_y \subseteq V \quad .$$

This implies $U \subseteq \ker(\varphi)$ and hence we get an induced R-homomorphism $f : R_x/U \to R_{xy}/V$. The set $\{\frac{1}{x^i} + U | i \in \mathbb{N}^+\}$ is a generating set for R_x/U and so we have

$$f \text{ is surjective} \iff \varphi \text{ is surjective} \iff \{\rho_1, \rho_2, \ldots\} \text{ generates } R_{xy}/V.$$

The set $\{\frac{1}{x^i y^j} + V | i, j \in \mathbb{N}^+\}$ generates R_{xy}/V. For $i \in \mathbb{N}^+$ we set

$$\mathcal{B}_i := \begin{pmatrix} b_1 & b_2 & b_3 & \ldots & b_i \\ b_2 & b_3 & b_4 & \ldots & b_{i+1} \\ \vdots & \vdots & \vdots & & \vdots \\ b_i & b_{i+1} & b_{i+2} & \ldots & b_{2i-1} \end{pmatrix}$$

Then we have for $i \in \mathbb{N}^+$:

$$(\rho_i, y\rho_{i+1}, \ldots, y^{i-1}\rho_{2i-1})^T = \mathcal{B}_i(\frac{1}{x^i y} + V, \frac{1}{x^{i-1}y^2} + V, \ldots, \frac{1}{xy^i} + V)^T.$$

If we choose $b_1, b_2, \ldots \in R$ in such a way that $\det \mathcal{B}_i \in R^*$ for all $i \in \mathbb{N}^+$ (which is possible, $\mathcal{B}$ consists only of ones and zeroes), then $\{\rho_1, \rho_2, \ldots\}$ generates R_{xy}/V.

From now on we assume that R is noetherian, we can use Čech cohomology to compute local cohomology. Thus, lemma 3.1.1 implies:

3.1.2 Theorem
Let R be a noetherian ring, M an R-module, $m \in \mathbb{N}^+, n \in \mathbb{N}, x_1, \ldots, x_m, y_1, \ldots, y_n \in R$. Then there exists an R-epimorphism

$$\mathrm{H}^m_{(x_1,\ldots,x_m)R}(M) \to \mathrm{H}^{m+n}_{(x_1,\ldots,x_m,y_1,\ldots,y_n)R}(M).$$

Proof:
Obviously it suffices to prove the statement for the case $M = R$. Using Čech cohomology to compute both local cohomology modules the statement follows immediately from lemma by induction on n.

By dualizing the surjection from the preceding theorem we get an injection. But, then, the set of associated prime ideals of the right-hand side is contained in the set of associated prime ideals of the left-hand side. This is the basic idea in the proof of statement (ii) in the following theorem (the same is true for (iii), (iv) and (v), as these statements follow from (ii), see the proof below for details):

3.1.3 Theorem
Let $(R, \mathfrak{m})$ be a noetherian local ring, $m \in \mathbb{N}^+, x_1, \ldots, x_m \in R$ and M a finitely generated R-module. Then the following statements hold:
(i) $\dim(M/\mathfrak{p}M) \geq m$ for every $\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^m_{(x_1,\ldots,x_m)R}(M)))$.
(ii) $\{\mathfrak{p} \in \mathrm{Supp}_R(M) | x_1, \ldots, x_m$ is part of a system of parameters of $R/\mathfrak{p}\} \subseteq \mathrm{Ass}_R(D(\mathrm{H}^m_{(x_1,\ldots,x_m)R}(M)))$.
(iii) $\mathrm{Ass}_R(D(\mathrm{H}^1_{xR}(R))) = \mathrm{Spec}(R) \setminus \mathfrak{V}(x)$ for every $x \in R$.
(iv) If $x_1, \ldots, x_m$ is part of a system of parameters of M, we have $\mathrm{Assh}(M) \subseteq \mathrm{Ass}_R(D(\mathrm{H}^m_{(x_1,\ldots,x_m)R}(M)))$; furthermore, if $m = \dim(M)$, equality holds: $\mathrm{Assh}(M) = \mathrm{Ass}_R(D(\mathrm{H}^{\dim(M)}_{\mathfrak{m}}(M)))$ (note that, by definition, $\mathrm{Assh}(M)$ consists of the associated prime ideals of M of highest dimension).

(v) If R is complete, $\mathfrak{p} \in \mathrm{Supp}_R(M)$ and $\dim(R/\mathfrak{p}) = m$, the equivalence

$$\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^m_{(x_1,\ldots,x_m)R}(M))) \iff x_1, \ldots, x_m \text{ is a system of parameters of } R/\mathfrak{p}$$

holds.

Proof:

We set $I := (x_1, \ldots, x_m)R$.

(i) Let $\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^m_I(M)))$. We conclude

$$0 \neq \mathrm{Hom}_R(R/\mathfrak{p}, D(\mathrm{H}^m_I(M))) = D(\mathrm{H}^m_I(M) \otimes_R (R/\mathfrak{p})) = D(\mathrm{H}^m_I(M/\mathfrak{p}M)).$$

Thus we have $\mathrm{H}^m_I(M/\mathfrak{p}M) \neq 0$ and statement (i) follows (note that it follows also from remark 1.2.1).

(ii) Let $\mathfrak{p} \in \mathrm{Supp}_R(M)$ such that $x_1, \ldots, x_m$ is part of a system of parameters of $R/\mathfrak{p}$. By completing $x_1, \ldots, x_m$ to a system of parameters of $M/\mathfrak{p}M$ and using theorem 3.1.2, we may assume that $x_1, \ldots, x_m$ is a system of parameters of $M/\mathfrak{p}M$. So we have $\dim M/\mathfrak{p}M{=}\dim(R/\mathfrak{p}) = m$. Therefore we get

$$\begin{aligned}\mathrm{Hom}_R(R/\mathfrak{p}, D(\mathrm{H}^m_I(M))) &= D(\mathrm{H}^m_I(M/\mathfrak{p}M)) \\ &= D(\mathrm{H}^m_{\mathfrak{m}}(M/\mathfrak{p}M))) \\ &\neq 0.\end{aligned}$$

On the other hand we have $\mathrm{Hom}_R(R/\mathfrak{q}, D(\mathrm{H}^m_I(M))) = 0$ for every prime ideal $\mathfrak{q}$ of R containing $\mathfrak{p}$ properly, by (i); statement (ii) follows.

(iii) Using (ii), it remains to show that $x \notin \mathfrak{p}$ holds for every $\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^1_{xR}(R)))$. As we have seen above, our hypothesis implies $\mathrm{H}^1_{xR}(R/\mathfrak{p}) \neq 0$. So we must have $x \notin \mathfrak{p}$.

(iv) The first statement follows from (ii) (note that, for every $\mathfrak{p} \in \mathrm{Assh}(M)$, $x_1, \ldots, x_m$ is part of a system of parameters of $R/\mathfrak{p}$, too) and then the second statement from (i).

(v) Let $\mathfrak{p} \in \mathrm{Supp}_R(M)$ such that $\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^m_I(M)))$. We have to show that $x_1, \ldots, x_m$ is a system of parameters of $M/\mathfrak{p}M$: $\mathrm{H}^m_I(M/\mathfrak{p}M) \neq 0$ implies $\mathrm{H}^m_I(R/\mathfrak{p}) \neq 0$. As R and hence $R/\mathfrak{p}$ are complete we may conclude from Hartshorne-Lichtenbaum vanishing (see, e. g. , [BS, 8.2.1] or theorem 6.1.4 for a reference on Hartshorne-Lichtenbaum vanishing) that $\dim(R/(I + \mathfrak{p})) = 0$, i. e. $x_1, \ldots, x_m$ is a system of parameters of $R/\mathfrak{p}$.

3.2 On the set $\mathrm{Ass}_R(D(\mathrm{H}^{\dim(R)-1}_I(R)))$

We prove a series of lemmas which we will need for the main results 3.2.6 and 3.2.7.

3.2.1 Lemma

Let $(S, \mathfrak{m})$ be a noetherian local complete Gorenstein ring of dimension $n+1$ (≥ 1) and $\mathfrak{P} \subseteq S$ a prime ideal of height n. Then

$$D(\mathrm{H}^n_{\mathfrak{P}}(S)) = \widehat{S_{\mathfrak{P}}}/S$$

holds canonically.

Proof:

Local duality over the Gorenstein ring S shows that there are natural isomorphisms

$$D(\mathrm{H}^n_{\mathfrak{P}}(S)) = D(\varinjlim_{l \in \mathbf{N}} \mathrm{Ext}^n_S(S/\mathfrak{P}^l, S)) = \varprojlim_{l \in \mathbf{N}} \mathrm{H}^1_{\mathfrak{m}}(S/\mathfrak{P}^l) \quad .$$

Take $y \in \mathfrak{m} \setminus \mathfrak{P}$. Now, $\sqrt{y(S/\mathfrak{P}^l)} = \mathfrak{m}(S/\mathfrak{P}^l)$ implies

$$\mathrm{H}^1_{\mathfrak{m}}(S/\mathfrak{P}^l) = \mathrm{H}^1_{y(S/\mathfrak{P}^l)}(S/\mathfrak{P}^l) = (S_y/\mathfrak{P}^l S_y)/(S/\mathfrak{P}^l) = (S_{\mathfrak{P}}/\mathfrak{P}^l S_{\mathfrak{P}})/(S/\mathfrak{P}^l)$$

and the statement follows by observing that the maps

$$(S_{\mathfrak{P}}/\mathfrak{P}^{l+1} S_{\mathfrak{P}})/(S/\mathfrak{P}^{l+1}) \to (S_{\mathfrak{P}}/\mathfrak{P}^l S_{\mathfrak{P}})/(S/\mathfrak{P}^l)$$

which we get from this, are the natural ones.

3.2.2 Lemma
Let $(R, \mathfrak{m})$ be a noetherian local complete domain and $I \subseteq R$ a prime ideal such that $\dim(R/I) = 1$. Then there exist a noetherian local complete regular ring S, a local homomorphism $S \xrightarrow{\rho} R$ and a prime ideal $\mathfrak{Q} \subseteq S$ such that R is finite as an S-module and such that

$$\mathrm{height}(\ker(\rho)) = 1, \dim(S/\mathfrak{Q}) = 1, \sqrt{\mathfrak{Q}R} = I, \ker(\rho) \subseteq \mathfrak{Q}$$

hold.
Proof:
Either R contains a field k or, if not, a coefficient ring (V, tV); choose $y_1, \ldots, y_n, y \in R$ such that

$$I = \sqrt{(y_1, \ldots, y_n)R}$$

and $y_1, \ldots, y_{n-1}, y$ is a system of parameters of R ($\dim(R) = n$); in the case of mixed characteristic we may take $y_1 := t$ if $t \in I$ and $y := t$ if $t \notin I$. Now we define a subring of R:

$$R_0 := k[[y, y_1, \ldots, y_n]]$$

(if R contains a field) resp.

$$R_0 := V[[y, y_2, \ldots, y_n]]$$

(if R contains no field and $t \in I$) resp.

$$R_0 := V[[y_1, \ldots, y_n]]$$

(if R contains no field and $t \notin I$). Furthermore we define a power series ring

$$S := k[[Y, Y_1, \ldots, Y_n]] \text{ resp. } V[[Y, Y_2, \ldots, Y_n]] \text{ resp. } V[[Y_1, \ldots, Y_n]]$$

(all capital letters denote variables) and it is clear how to define a surjection $S \xrightarrow{\rho} R_0 (\subseteq R)$. We set

$$\mathfrak{Q} := (Y_1, \ldots, Y_n)S \text{ resp. } (t, Y_2, \ldots, Y_n)S \text{ resp. } (Y_1, \ldots, Y_n)S,$$

where in all three cases we have $\ker(\rho) \subseteq \mathfrak{Q}$ because of $y \notin I$. All other statements are obvious now.

3.2.3 Lemma
Let R be a noetherian ring.
(i) Let $\mathfrak{P}$ be a prime ideal of R which is not maximal. Then the equivalence

$$R_{\mathfrak{P}} = \widehat{R_{\mathfrak{P}}} \iff \mathfrak{P} \text{ is minimal in } \mathrm{Spec}(R)$$

holds.
(ii) Assume that R is local (and noetherian) and that all prime ideals associated to R are minimal in $\operatorname{Spec}(R)$. Then $\operatorname{Ass}_R(\hat{R}/R) \subseteq \operatorname{Ass}(R)$ holds. In particular if R is a non-complete (local) domain (i. e. if $R \subsetneq \hat{R}$),

$$\operatorname{Ass}_R(\hat{R}/R) = \{0\}$$

holds.
Proof:
(i) The implication $\Leftarrow$ is clear as every zero-dimensional local noetherian ring is complete. We assume there exists a prime ideal P of R which is neither minimal nor maximal in $\operatorname{Spec}(R)$ and such that $R_P = \widehat{R_P}$. PR_P is not minimal in $\operatorname{Spec}(R)$. Choose $Q, Q' \in \operatorname{Spec}(R)$ such that $Q' \subsetneq P \subsetneq Q$ and such that $\dim(R_Q/PR_Q) = 1$. We set $\mathfrak{R} := R_Q/Q'R_Q$ and $\mathfrak{P} := PR_Q/Q'R_Q \in \operatorname{Spec}(\mathfrak{R})$ and we get

$$\mathfrak{R}_P = R_P/Q'R_P = \widehat{R_P}/Q'\widehat{R_P} = \widehat{R_P/Q'R_P} = \widehat{\mathfrak{R}_P}.$$

So we may assume that R is a local domain and $\dim(R/P) = 1$.
Take $y \in \mathfrak{m} \setminus P$. Assume that for some $n \in \mathbb{N}$

$$P^{(n)} \subseteq P^{(n+1)} + yR$$

holds ($P^{(n)} := P^n R_P \cap R$ is a P-primary ideal of R such that $P^{(n)} R_P = P^n R_P$, $P^{(n)}$ is the so-called n-th symbolic power of P). It would follow that

$$P^{(n)} = P^{(n)} \cap (P^{(n+1)} + yR) = P^{(n+1)} + (P^{(n)} \cap yR) = P^{(n+1)} + yP^{(n)}$$

(the last equality follows, because $\mathfrak{P}^{(n)}$ is $\mathfrak{P}$-primary) and then $P^{(n)} = P^{(n+1)}$, by the lemma of Nakayama (see, e. g., [Ma, Theorem 2.2] for the lemma of Nakayama). Again by the lemma of Nakayama, this would imply $P^n R_P = 0$ and so PR_P would be minimal in $\operatorname{Spec}(R_P)$. We conclude that for every $n \in \mathbb{N}$

$$P^{(n)} \not\subseteq P^{(n+1)} + yR$$

holds. For every $n \in \mathbb{N}$ we choose $x_n \in P^{(n)} \setminus (P^{(n+1)} + yR)$ and define (for every $n \in \mathbb{N}^+$)

$$\xi_n := \sum_{i=0}^{n-1} \frac{x_i}{y^{(i+1)^2}} \in R_P.$$

Because of

$$\xi_{n+1} - \xi_n = \frac{x_n}{y^{(n+1)^2}} \in P^n R_P$$

(for every n), we have

$$(\xi_n + P^n R_P)_{n \in \mathbb{N}^+} \in \widehat{R_P} = R_P.$$

Therefore, there exists $\xi \in R_P$ such that

$$(\xi + P^n R_P)_{n \in \mathbb{N}^+} = (\xi_n + P^n R_P)_{n \in \mathbb{N}^+},$$

i. e.

$$\xi - \xi_n \in P^n R_P$$

holds for every $n \in \mathbb{N}^+$.
Write $\xi = \frac{a}{b}$, where $a \in R, b \in R \setminus P$. The ideal $P + bR$ of R is either R or $\mathfrak{m}$-primary, so there exist $p \in \mathbb{N}^+$ and $c \in R$ such that $y^p - bc \in P$; it follows that

$$y^{pn} - bc_n \in P^n,$$

where

$$c_n := b^{-1}(y^{pn} - (y^p - bc)^n) \in R$$

(note that $y^{pn} - (y^p - bc)^n$ is divisible by b in R) and we conclude that

$$\xi - \frac{ac_n}{y^{pn}} = \frac{ay^{pn} - abc_n}{by^{pn}} = \frac{a(y^p - bc)^n}{by^{pn}} \in P^n R_P$$

for every $n \in \mathbb{N}^+$. We get

$$\xi_n - \frac{ac_n}{y^{pn}} = \xi - \frac{ac_n}{y^{pn}} - (\xi - \xi_n) \in P^n R_P$$

for every $n \in \mathbb{N}^+$. From this we get (for $n > p$) after multiplication by y^{n^2} that

$$\sum_{i=0}^{n-1} x_i y^{n^2-(i+1)^2} - ac_n y^{n(n-p)} \in P^{(n)}$$

and in particular $x_{n-1} \in P^{(n)} + yR$ which is a contradiction.
(ii) We only have to prove the first statement, the second one follows from it immediately; Let P be an arbitrary element of $\operatorname{Spec}(R) \setminus \operatorname{Ass}(R)$; We conclude $\operatorname{Hom}_R(R/P, R) = 0$ and hence also $\operatorname{Hom}_R(R/P, \hat{R}) = 0$ (because P contains an element which operates injectively on R and $\hat{R}$ is flat over R). Thus the short exact sequence

$$0 \to R \stackrel{\subseteq}{\to} \hat{R} \to \hat{R}/R \to 0$$

induces an exact sequence

$$0 \to \operatorname{Hom}_R(R/P, \hat{R}/R) \to \operatorname{Ext}^1_R(R/P, R) \stackrel{\varphi}{\to} \operatorname{Ext}^1_R(R/P, \hat{R}).$$

By our hypothesis there exists $x \in P$ such that $x \notin Q$ for all $Q \in \operatorname{Ass}(R)$. We get short exact sequences

$$0 \to R \stackrel{x}{\to} R \to R/xR \to 0$$

and

$$0 \to \hat{R} \stackrel{x}{\to} \hat{R} \to \hat{R}/x\hat{R} \to 0.$$

Because of $x \in P$ a commutative diagram with exact rows is induced:

$$\begin{array}{ccccccc} 0 & \to & \operatorname{Hom}_R(R/P, R/xR) & \to & \operatorname{Ext}^1_R(R/P, R) & \to & 0 \\ & & \downarrow \psi & & \downarrow \varphi & & \\ 0 & \to & \operatorname{Hom}_R(R/P, \hat{R}/x\hat{R}) & \to & \operatorname{Ext}^1_R(R/P, \hat{R}) & \to & 0. \end{array}$$

ψ is injective as $R/xR \subseteq \widehat{R/xR} = \hat{R}/x\hat{R}$. Therefore, φ is injective which implies that $\operatorname{Hom}_R(R/P, \hat{R}/R) = 0$, i. e. $P \notin \operatorname{Ass}_R(\hat{R}/R)$.

The following result is a special case of both 3.2.6 and 3.2.7.

3.2.4 Theorem
Let $(R, \mathfrak{m})$ be a d-dimensional local noetherian complete domain, where $d \geq 2$; let P be a prime ideal of R such that $\dim(R/P) = 1$. Then

$$\{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^{d-1}_P(R)))$$

holds.
Proof:
We apply lemma 3.2.2, set $R_0 := \mathrm{im}(\rho)$ and consider the ideal $\mathfrak{Q}$ from lemma 3.2.2 as an ideal of R_0. Because of lemma 3.2.2, R_0 is a complete intersection, in particular it is Gorenstein. By $\mathfrak{m}_0$ we denote the maximal ideal of R_0. R is finite over R_0 and so we have $D_{R_0}(R) = \mathrm{Hom}_{R_0}(R, \mathrm{E}_{R_0}(R_0/\mathfrak{m}_0))) = \mathrm{E}_R(R/\mathfrak{m}) = D_R(R)$, which implies $D_{R_0}(M) = D_R(M)$ for every R-module M. On the other hand the functor $\mathrm{H}^{d-1}_{\mathfrak{Q}}(\_)$ is right exact by Hartshorne-Lichtenbaum vanishing; in particular we have

$$D_R(\mathrm{H}^{d-1}_P(R)) = D_{R_0}(\mathrm{H}^{d-1}_{\mathfrak{Q}}(R_0) \otimes_{R_0} R) = \mathrm{Hom}_{R_0}(R, D_{R_0}(\mathrm{H}^{d-1}_{\mathfrak{Q}}(R_0)))$$

and so every R_0-monomorphism $R_0 \to D_{R_0}(\mathrm{H}^{d-1}_{\mathfrak{Q}}(R_0))$ induces an R-monomorphism

$$\mathrm{Hom}_{R_0}(R, R_0) \to D_R(\mathrm{H}^{d-1}_{\mathfrak{P}}(R)) \quad .$$

It is easy to see that $\{0\} \in \mathrm{Ass}_{R_0}(\mathrm{Hom}_{R_0}(R, R_0))$ holds (e. g. by localizing) and thus it suffices to show $\{0\} \in \mathrm{Ass}_{R_0}(D_{R_0}(\mathrm{H}^{d-1}_{\mathfrak{Q}}(R_0)))$, i. e. we may assume R_0 is Gorenstein. Now, by lemma 3.2.1, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & R & \overset{\subseteq}{\to} & \widehat{R_P} & \to & D(\mathrm{H}^{d-1}_P(R)) & \to & 0 \\ & & \downarrow\subseteq & & \downarrow= & & & & \\ 0 & \to & R_P & \overset{\subseteq}{\to} & \widehat{R_P} & \to & \widehat{R_P}/R_p & \to & 0. \end{array}$$

This diagram induces an epimorphism

$$D(\mathrm{H}^{d-1}_P(R)) \to \widehat{R_P}/R_P.$$

By lemma 3.2.3 (i) we have $\widehat{R_P}/R_P \neq 0$ and it follows from lemma 3.2.3 (ii) that $(\widehat{R_P}/R_P) \otimes_R (Q(R)) \neq 0$. Thus we have $D(\mathrm{H}^{d-1}_P(R)) \otimes_R Q(R) \neq 0$ by the above epimorphism, which is equivalent to the statement of the theorem.

3.2.5 Lemma
Let $(R, \mathfrak{m})$ be a noetherian local complete domain, $d := \dim(R) \geq 1$ and $J \subseteq R$ an ideal of R such that $\dim(R/J) = 1$. Then

$$\mathrm{Assh}(R) \cap \mathrm{Ass}_R(D(\mathrm{H}^{d-1}_J(R))) = \{Q \in \mathrm{Assh}(R) | \dim(R/(J+Q)) \geq 1\}$$

holds. In particular, if $\mathrm{H}^d_J(R) = 0$,

$$\mathrm{Assh}(D(\mathrm{H}^{d-1}_J(R))) = \mathrm{Assh}(R)$$

holds.
Proof:

The second statement follows from the first one by Hartshorne-Lichtenbaum vanishing. First, we prove the statement in the special case where $\dim(R/(J+Q)) \geq 1$ for all $Q \in \operatorname{Assh}(R)$; then we will reduce the general to this special situation. Now, in the special case, it suffices to show the inclusion "$\supseteq$": By Hartshorne-Lichtenbaum vanishing we have $\mathrm{H}_J^d(R) = 0$, i. e. H_J^{d-1} is right exact. Let $Q \in \operatorname{Assh}(R)$ be arbitrary. The canonical epimorphism $R \to R/Q =: \overline{R}$ induces a monomorphism

$$D_{\overline{R}}(\mathrm{H}_{J\overline{R}}^{d-1}(\overline{R})) = D_R(\mathrm{H}_J^{d-1}(\overline{R})) \to D(\mathrm{H}_J^{d-1}(R)).$$

If $\{0\} \in \operatorname{Ass}_{\overline{R}}(D_{\overline{R}}(\mathrm{H}_{J\overline{R}}^{d-1}(\overline{R})))$ then $Q \in \operatorname{Ass}_R(D_R(\mathrm{H}_J^{d-1}(R)))$, and so we may assume that R is a domain. If we can write $J = J_1 \cap J_2$ with non-$\mathfrak{m}$-primary ideals J_1, J_2 of R such that $J_1 + J_2$ is $\mathfrak{m}$-primary then, because of $\mathrm{H}_{J_1}^d(R) = \mathrm{H}_{J_2}^d(R) = 0$ (Hartshorne-Lichtenbaum vanishing), a Mayer-Vietoris sequence argument shows the existence of an epimorphism

$$\mathrm{H}_J^{d-1}(R) \to \mathrm{H}_{J_1+J_2}^d(R) = \mathrm{H}_{\mathfrak{m}}^d(R).$$

But then theorem 3.1.3 (iv) implies that

$$\{0\} = \operatorname{Assh}(R) = \operatorname{Ass}_R(D(\mathrm{H}_{\mathfrak{m}}^d(R))) \subseteq \operatorname{Ass}_R(D(\mathrm{H}_J^{d-1}(R))).$$

If there is no such decomposition $J = J_1 \cap J_2$ of J we may assume that J is a prime ideal; but then the statement follows from theorem 3.2.4. Now we turn to the general case, i. e. we assume there is a $Q \in \operatorname{Assh}(R)$ such that $\dim(R/(J+Q)) = 0$. We define $U(R)$ to be the intersection of all Q'-primary components of a primary decomposition of the zero ideal in R for all $Q' \in \operatorname{Assh}(R)$. Apparently we have $\operatorname{Ass}_R(R/U(R)) = \operatorname{Assh}(R)$ and $\dim(U(R)) < d$. Because of the latter fact the short exact sequence $0 \to U(R) \overset{\subseteq}{\to} R \to R/U(R) \to 0$ induces an exact sequence

$$0 \to D(\mathrm{H}_J^{d-1}(R/U(R))) \to D(\mathrm{H}_J^{d-1}(R)) \to D(\mathrm{H}_J^{d-1}(U(R))).$$

Trivially $\dim_R(\operatorname{Supp}_R(\mathrm{H}_J^{d-1}(U(R)))) \leq d-1$ holds. By considering $R/U(R)$ rather then R we may assume that $\operatorname{Ass}_R(R) = \operatorname{Assh}(R)$. We write $0 = I' \cap I''$ with ideals I', I'' of R such that $\operatorname{Ass}_R(R) = \operatorname{Ass}_R(R/I') \cup \operatorname{Ass}_R(R/I'')$ and $\dim(R/(J+Q')) \geq 1$ for all $Q' \in \operatorname{Ass}_R(R/I')$ and $\dim(R/(J+Q'')) = 0$ for all $Q'' \in \operatorname{Ass}_R(R/I''))$. It follows that $\dim(R/(J+I'')) = 0$. By using a Mayer-Vietoris argument and the facts that $\mathrm{H}_J^d(R/I') = 0$ (Hartshorne-Lichtenbaum vanishing) and $\mathrm{H}_J^i(R/I'') = \mathrm{H}_{\mathfrak{m}}^i(R/I'')$ for all $i \in \mathbb{N}$ we get a short exact sequence

$$D(\mathrm{H}_{\mathfrak{m}}^{d-1}(R/I'+I'')) \to D(\mathrm{H}_J^{d-1}(R/I')) \oplus D(\mathrm{H}_{\mathfrak{m}}^{d-1}(R/I'')) \to$$
$$\to D(\mathrm{H}_J^{d-1}(R)) \to D(\mathrm{H}_{\mathfrak{m}}^{d-2}(R/(I'+I''))).$$

It is clear that we have

$$\dim_R(\operatorname{Supp}_R(D(\mathrm{H}_{\mathfrak{m}}^{d-1}(R/(I'+I''))))) \leq d-1$$

and

$$\dim_R(\operatorname{Supp}_R(D(\mathrm{H}_{\mathfrak{m}}^{d-2}(R/(I'+I''))))) \leq d-1.$$

R is complete and so we can use local duality to conclude that

$$\dim_R(\operatorname{Supp}_R(D(\mathrm{H}_{\mathfrak{m}}^{d-1}(R/I'')))) \leq d-1.$$

Thus we get, by what is already shown,

$$\operatorname{Assh}(R) \cap \operatorname{Ass}_R(D(\mathrm{H}_J^{d-1}(R))) = \{Q \in \operatorname{Ass}_R(D(\mathrm{H}_J^{d-1}(R/I'))) | \dim(R/Q) = d\} = \operatorname{Assh}(R/I').$$

The following theorems 3.2.6 (where R is not necessarily complete) and 3.2.7 (where R will be complete) contain the main results.

3.2.6 Theorem
Let $(R, \mathfrak{m})$ be a d-dimensional noetherian local ring and $J \subseteq R$ an ideal such that $\dim(R/J) = 1$ and $\mathrm{H}^d_J(R) = 0$. Then

$$\mathrm{Assh}(R) = \mathrm{Assh}(D(\mathrm{H}^{d-1}_J(R)))$$

holds.
Proof:
One has $\mathrm{H}^d_{J\hat{R}}(\hat{R}) = \mathrm{H}^d_J(R) \otimes_R \hat{R} = 0$ and

$$\begin{aligned} D_{\hat{R}}(\mathrm{H}^{d-1}_{J\hat{R}}(\hat{R})) &= D_{\hat{R}}(\mathrm{H}^{d-1}_J(R) \otimes \hat{R}) \\ &= \mathrm{Hom}_R(\mathrm{H}^{d-1}_J(R), D_{\hat{R}}(\hat{R})) \\ &= D_R(\mathrm{H}^{d-1}_J(R)) \end{aligned}$$

Therefore, every $\hat{R}$-monomorphism $\varphi : \hat{R}/\mathfrak{P} \to D_{\hat{R}}(\mathrm{H}^{d-1}_{J\hat{R}}(\hat{R}))$, where $\mathfrak{P}$ is a prime ideal of $\hat{R}$, induces an R-monomorphism $\hat{R}/\mathfrak{P} \to D_R(\mathrm{H}^{d-1}_J(R))$. On the other hand we have a R-monomorphism $R/\mathfrak{p} \to \hat{R}/\mathfrak{P}$, where $\mathfrak{p} := \mathfrak{P} \cap R$. Composition of these monomorphisms gives us a monomorphism

$$R/\mathfrak{p} \to D_R(\mathrm{H}^{d-1}_J(R)).$$

Because of $\mathrm{Assh}(R) = \{\mathfrak{P} \cap R | \mathfrak{P} \in \mathrm{Assh}(\hat{R})\}$ we may assume that R is complete. But then the statement follows from lemma 3.2.5.

3.2.7 Theorem
Let R be a d-dimensional local complete ring and $J \subseteq R$ an ideal such that $\dim(R/J) = 1$ and $\mathrm{H}^d_J(R) = 0$. Then

$$\mathrm{Ass}_R(D(\mathrm{H}^{d-1}_J(R)) = \{P \in \mathrm{Spec}(R) | \dim(R/P) = d-1, \dim(R/(P+J)) = 0\} \cup \mathrm{Assh}(R)$$

holds.
Proof:
Let $P \in \mathrm{Spec}(R)$. If $\dim(R/P) \leq d-2$ we have

$$\mathrm{Hom}_R(R/P, D(\mathrm{H}^{d-1}_J(R))) = D(\mathrm{H}^{d-1}_J(R/P)) = 0$$

and hence $P \notin \mathrm{Ass}_R(D(\mathrm{H}^{d-1}_J(R)))$. If $\dim(R/P) = 1$ then (set $\overline{R} := R/P$):

$$\mathrm{Hom}_R(R/P, D(\mathrm{H}^{d-1}_J(R))) = D(\mathrm{H}^{d-1}_J(R/P)) = D(\mathrm{H}^{d-1}_{J\overline{R}}(\overline{R})).$$

R is complete and so, by Hartshorne-Lichtenbaum vanishing, the equivalence

$$\mathrm{H}^{d-1}_{J\overline{R}}(\overline{R}) \neq 0 \iff \dim(\overline{R}/J\overline{R}) = 0$$

holds. On the other hand we have $\overline{R}/J\overline{R} = R/(P+J)$ and, therefore

$$\{P \in \mathrm{Ass}_R(D(\mathrm{H}^{d-1}_J(R))) | \dim(R/P) = d-1\} =$$

$$= \{P \in \mathrm{Spec}(R) | \dim(R/P) = d-1, \dim(R/(P+J)) = 0\} \quad .$$

Now the statement follows from lemma 3.2.5.

4 The regular case and how to reduce to it

By "regular case" we mean the following situation: Let k be a field, $R = k[[X_1, \dots, X_n]]$ a power series algebra over k in n variables and I the ideal $(X_1, \dots, X_h)R$ of R $(1 \leq h \leq n)$. We are interested in the associated prime ideals of

$$D := D(\mathrm{H}^h_I(R)) \quad .$$

In the first subsection we will demonstrate how one can reduce conjecture (*) to the regular case, in subsection 4.2 we present results on $\mathrm{Ass}_R(D)$ for general h; subsection 4.3 concentrates on the case $h = n-2$, which is in some sense the "first" interesting case.

4.1 Reductions to the regular case

Suppose that $(R, \mathfrak{m})$ is a noetherian local ring. After completing R, we can write R as a quotient of a regular local ring S; on the other hand we can find a regular local subring S of R such that R is module-finite over S. We will use both methods to reduce to the regular case, i. e. to make facts about $\mathrm{Ass}_S(D_S)$ into facts about $\mathrm{Ass}_R(D_R)$ (D_S and D_R stand for the Matlis duals of local cohomology modules of S resp. R), see remark 4.1.1 and theorem 4.1.2 for details.

4.1.1 Remark

Suppose that $(R, \mathfrak{m})$ is a noetherian local equicharacteristic domain, $i \geq 1$ and $x_1, \dots, x_i$ is a sequence in R such that $\mathrm{H}^i_{(x_1,\dots,x_i)R}(R) \neq 0$. Suppose furthermore, that one wants to show $\{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\dots,x_i)R}(R)))$ (that is conjecture (*) in the equicharacteristic case). W. l. o. g. one can replace R by $\hat{R}/\mathfrak{p}_0$, where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R and $\mathfrak{p}_0 \in \mathrm{Spec}(\hat{R})$ is lying over the zero ideal of R (because then

$$\begin{aligned}
D_{\hat{R}/\mathfrak{p}_0}(\mathrm{H}^i_{(x_1,\dots,x_i)(\hat{R}/\mathfrak{p}_0)}(\hat{R}/\mathfrak{p}_0)) &= \mathrm{Hom}_{\hat{R}/\mathfrak{p}_0}(\mathrm{H}^i_{(x_1,\dots,x_i)(\hat{R})}(\hat{R}) \otimes_{\hat{R}} (\hat{R}/\mathfrak{p}_0), \mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{p}_0, \mathrm{E}_{\hat{R}}(k)))) \\
&= \mathrm{Hom}_{\hat{R}}(\mathrm{H}^i_{(x_1,\dots,x_i)(\hat{R})}, \mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{p}_0, \mathrm{E}_{\hat{R}}(k))) \\
&= \mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{p}_0, \mathrm{Hom}_{\hat{R}}(\mathrm{H}^i_{(x_1,\dots,x_i)\hat{R}}(\hat{R}), \mathrm{E}_{\hat{R}}(k))) \\
&= \mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{p}_0, \mathrm{Hom}_{\hat{R}}(\mathrm{H}^i_{(x_1,\dots,x_i)\hat{R}}(\hat{R}), \mathrm{E}_R(k))) \\
&= \mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{p}_0, D_R(\mathrm{H}^i_R(\mathrm{H}^i_{(x_1,\dots,x_i)R}(R))))
\end{aligned}$$

contains en element d with $\hat{R}$-annihilator $\mathfrak{p}_0$, i. e. with R-annihilator $\mathfrak{p}_0 \cap R = 0$; but d is naturally an element of $D_R(\mathrm{H}^i_{(x_1,\dots,x_i)R}(R))$), and so we may assume that $(R, \mathfrak{m})$ is a noetherian local equicharacteristic complete domain. Let k be a coefficient field of R. Now if we use a surjective k-algebra homomorphism $k[[X_1, \dots, X_n]] \to R$ ($k[[X_1, \dots, X_n]]$ is a power series algebra over k in n variables $X_1, \dots, X_n$) mapping $X_1, \dots, X_n$ to $x_1, \dots x_n$, respectively, we can reduce to the following problem (note that, below, $\mathfrak{p}$ corresponds to the zero ideal of R):
If $R = k[[X_1, \dots, X_n]]$ is a power series ring over a field k in n variables $X_1, \dots, X_n$, $1 \leq i \leq n$, $\mathfrak{q} \in \mathrm{Ass}_R(D(\mathrm{H}^i_{(X_1,\dots,X_i)R}(R)))$, $\mathfrak{p} \in \mathrm{Spec}(R)$, $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^i_{(X_1,\dots,X_i)R}(R)))$ holds (that is: The set $\mathrm{Ass}_R(D(\mathrm{H}^i_{(X_1,\dots,X_i)R}(R)))$ is stable under generalization).
Thus we have reduced conjecture (*) (in the equicharacteristic case) to the preceding statement, a similar reduction is possible in the case of mixed characteristic.

4.1.2 Theorem

Let $(R, \mathfrak{m})$ be a noetherian local complete ring with coefficient field $k \subseteq R$, $l \in \mathbb{N}^+$ and $x_1, \ldots, x_l \in R$ a part of a system of parameters of R. Set $I := (x_1, \ldots, x_l)R$. Let $x_{l+1}, \ldots, x_d \in R$ be such that $x_1, \ldots, x_d$ is a system of parameters of R. Denote by R_0 the (regular) subring $k[[x_1, \ldots, x_d]]$ of R. Then if $\mathrm{Ass}_{R_0}(D(\mathrm{H}^l_{(x_1,\ldots,x_l)R_0}(R_0)))$ is stable under generalization, $\mathrm{Ass}_R(D(\mathrm{H}^l_I(R)))$ is also stable under generalization.

Proof:

Set $X := \mathrm{Ass}_R(D(\mathrm{H}^m_I(R)))$ and let $\mathfrak{p}_1 \in \mathrm{Spec}(R), \mathfrak{p} \in X, \mathfrak{p}_1 \subseteq \mathfrak{p}$. We have to show $\mathfrak{p}_1 \in X$. The hypothesis on $\mathfrak{p}$ implies

$$0 \neq \mathrm{H}^l_I(R/\mathfrak{p}) = \mathrm{H}^l_{(x_1,\ldots,x_m)R_0}(R_0/\mathfrak{p} \cap R_0) \otimes_{R_0} R.$$

But $\mathrm{Ass}_{R_0}(D(\mathrm{H}^l_{(x_1,\ldots,x_l)R_0}(R_0)))$ is stable under generalization and so by using Matlis duality we first conclude $\mathfrak{p} \cap R_0 \in \mathrm{Ass}_{R_0}(D(\mathrm{H}^l_{(x_1,\ldots,x_l)R_0}(R_0)))$ and then, by using the stableness hypothesis again, $\mathfrak{p}_1 \cap R_0 \in \mathrm{Ass}_{R_0}(D(\mathrm{H}^l_{(x_1,\ldots,x_l)R_0}(R_0)))$. Now the existence of an R_0-linear injection $R_0/\mathfrak{p}_1 \cap R_0 \to D(\mathrm{H}^l_{(x_1,\ldots,x_l)R_0}(R_0))$ implies the existence of an R-linear injection

$$\begin{aligned} \mathrm{Hom}_{R_0}(R, R_0/\mathfrak{p}_1 \cap R_0) &\to \mathrm{Hom}_{R_0}(R, D(\mathrm{H}^l_{(x_1,\ldots,x_l)}(R_0))) \\ &= \mathrm{Hom}_{R_0}(\mathrm{H}^l_{(x_1,\ldots,x_l)R_0}(R_0), (\mathrm{Hom}_{R_0}(R, \mathrm{E}_{R_0}(k)))) \\ &= D(\mathrm{H}^l_I(R)), \end{aligned}$$

where the last equality follows from the fact $\mathrm{Hom}_{R_0}(R, \mathrm{E}_{R_0}(k)) = \mathrm{E}_R(k)$. Thus it is sufficient to show

$$\mathfrak{p}_1 \in \mathrm{Ass}_R(\mathrm{Hom}_{R_0}(R, R_0/\mathfrak{p}_1 \cap R_0)) = \mathrm{Hom}_{R_0/\mathfrak{p}_1 \cap R_0}(R/(\mathfrak{p}_1 \cap R_0)R, R_0/\mathfrak{p}_1 \cap R_0) \quad .$$

But R is finite as R_0-module and so $\mathrm{Hom}_{R_0/\mathfrak{p}_1 \cap R_0}(R/\mathfrak{p}_1, R_0/\mathfrak{p}_1 \cap R_0) \neq 0$; on the other hand $\mathfrak{p}_1$ is minimal in the support of $R/(\mathfrak{p}_1 \cap R_0)R$ and so, combining these facts, the statement of theorem 4.1.2 follows.

4.2 Results in the general case, i. e. h is arbitrary

We collect some properties of $\mathrm{Ass}_R(D)$ in the regular case; note that R does not have to contain a field:

4.2.1 Theorem

Let $(R, \mathfrak{m})$ be a noetherian local complete regular ring. Let $X_1, \ldots, X_n$ be a regular system of parameters of R, $n = \dim(R)$. Set $I := (X_1, \ldots, X_h)R$ for some $h \in \{1, \ldots, n\}$. Set $D := D(\mathrm{H}^h_I(R))$.

(i) For $h = n$ one has

$$\mathrm{Ass}_R(D) = \{\{0\}\} \quad .$$

(ii) For $h = n - 1$ one has

$$\mathrm{Ass}_R(D) = \{\{0\}\} \cup \{pR | p \in R \text{ prime element}, p \notin I\} \quad .$$

(iii) For general h the following statements hold:

(α) For every $\mathfrak{p} \in \mathrm{Spec}(R)$ the implication

$$\mathfrak{p} \in \mathrm{Ass}_R(D) \Longrightarrow \mathrm{height}(\mathfrak{p}) \leq n - h$$

holds.
(β) For every $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\operatorname{height}(\mathfrak{p}) = n - h$ one has the equivalence

$$\mathfrak{p} \in \operatorname{Ass}_R(D) \iff I + \mathfrak{p} \text{ is } \mathfrak{m}\text{-primary} \quad .$$

(γ) Every $f \in I \setminus \mathfrak{m}I$ is contained in no $\mathfrak{p} \in \operatorname{Ass}_R(D)$; in particular, if $f = p$ is a prime element of R (such that $p \in I \setminus \mathfrak{m}I$), one has

$$pR \notin \operatorname{Ass}_R(D) \quad .$$

(δ) If $p \in R$ is a prime element such that $p \notin I$ then

$$pR \in \operatorname{Ass}_R(D)$$

holds.
Proof:
(i) follows from $D(\mathrm{H}^n_{\mathfrak{m}}(R)) = R$, (ii) from theorem 3.2.7. (iii) (α) and (β) follow from theorem 3.1.3 (i) resp. (v). (iii) (γ): In the given situation one has

$$\operatorname{Hom}_R(R/fR, D) = D(\mathrm{H}^h_I(R)/f\,\mathrm{H}^h_I(R)) = D(\mathrm{H}^h_{I(R/fR)}(R/fR))$$

and $\mathrm{H}^h_{I(R/fR)}(R/fR) = 0$, because f is a minimal generator of I, i. e. $I(R/fR)$ can be generated by $h-1$ elements; thus multiplication by f is injective on D, the statement follows. (iii) (δ) follows from theorem 3.1.3 (ii).

4.2.2 Remark
In the situation of theorem 4.2.1, the largest h, for which we cannot completely determine $\operatorname{Ass}_R(D)$, is $h = n - 2$; the theorem leaves open the question which prime ideals $\mathfrak{p} = pR$, $p \in R$ a prime element, $p \in I$, are associated to R. In the next subsection we will concentrate on the case $h = n - 2$. We will give a partial answer to this open question (see corollary 4.3.1) and we will see (remarks 4.3.2 (i) and (ii)) that both $pR \in \operatorname{Ass}_R(D)$ and $pR \notin \operatorname{Ass}_R(D)$ can occur (both in the special case $p \in I$, $h = n - 2$).

4.2.3 Theorem
Let $(R_0, \mathfrak{m}_0)$ be a noetherian local complete equicharacteristic ring, let $\dim(R_0) = n - 1$, $k \subseteq R_0$ a coefficient field of R_0, $1 \leq h \leq n$. Let $x_1, \dots, x_n$ be elements of R_0 such that $\sqrt{(x_1, \dots, x_n)R_0} = \mathfrak{m}_0$. Set $I_0 := (x_1, \dots, x_h)R_0$. Let $R := k[[X_1, \dots, X_n]]$ be a power series algebra over k in the variables $X_1, \dots, X_n$, $I := (X_1, \dots, X_h)R$. Then the k-algebra homomorphism $R \to R_0$ determined by $X_i \mapsto x_i$ ($i = 1, \dots n$) induces a module-finite ring map $\iota : R/fR \to R_0$ for some prime element $f \in R$. We set

$$D := D(\mathrm{H}^h_I(R)) \quad .$$

Then
(i) D has an associated prime which contains f if and only if $\mathrm{H}^h_{I_0}(R_0) \neq 0$.
Furthermore if R_0 is regular and $\operatorname{height}(I_0) < h$, the following statements hold:
(ii) D has no associated prime ideal which contains f and has height $n - h$.
(iii) If $\mathrm{H}^h_{I_0}(R_0) \neq 0$, ($f$ is contained in an associated prime of D and) every maximal element $\mathfrak{q}$ of $\operatorname{Ass}_R(D)$ containing f has $\dim(R/\mathfrak{q}) > h$; we will see below (remark 4.3.2 (ii)) that this situation really occurs and,

therefore, it is in general not true that all maximal elements of $\mathrm{Ass}_R(D(\mathrm{H}^h_I(R)))$ have dimension h; note that his was conjecture (+) in [HS1, section 0] (see also remark 1.2.4 for more details on this).
Proof:
(i) follows from

$$\begin{aligned}\exists_{\mathfrak{p}\in \mathrm{Ass}_R(D)} f \in \mathfrak{p} &\iff \mathrm{Hom}_R(R/fR, D) \neq 0 \\ &\iff D(\mathrm{H}^h_I(R)/f\,\mathrm{H}^h_I(R)) \neq 0 \\ &\iff D(\mathrm{H}^h_I(R/fR)) \neq 0 \\ &\iff \mathrm{H}^h_I(R/fR) \neq 0 \\ &\iff \mathrm{H}^h_{I_0}(R_0) \neq 0 \quad .\end{aligned}$$

Note that, for the last equivalence, we use the fact that via ι R_0 is a finite and torsion-free R/fR-module.
From now on we assume, in addition, that R_0 is regular and that $\mathrm{height}(I_0) < h$.
(ii) We assume, to the contrary, that there is a prime ideal $\mathfrak{p} \in \mathrm{Ass}_R(D)$ such that $\mathrm{height}(\mathfrak{p}) = n-h$ and such that $f \in \mathfrak{p}$: R_0 is module-finite over R/fR, and so there exists $\mathfrak{q} \in \mathrm{Spec}(R_0)$ such that $\mathfrak{q} \cap (R/fR) = \mathfrak{p}/fR$. But now $\mathfrak{q} \cap R = \mathfrak{p}$ implies

$$\mathrm{height}(\mathfrak{q}) = (n-1) - \dim(R_0/\mathfrak{q}) = (n-1) - \dim(R/\mathfrak{p}) = \mathrm{height}(\mathfrak{p}) - 1 = n - h - 1$$

and therefore, from $\mathrm{height}(I_0) < h$, we conclude

$$\mathrm{height}(I_0 + \mathfrak{q}) < n - 1 \quad ,$$

which means that $(I_0 + \mathfrak{q})/\mathfrak{q}$ is not $\mathfrak{m}_0/\mathfrak{q}$-primary in $R_0/\mathfrak{q}$. Hence, by Hartshorne-Lichtenbaum vanishing,

$$\mathrm{H}^h_{I_0}(R_0/\mathfrak{q}) = 0 \quad .$$

But $R_0/\mathfrak{q}$ is a torsion-free finite $R/\mathfrak{p}$-module, and so the last vanishing result implies

$$\mathrm{H}^h_I(R/\mathfrak{p}) = 0 \quad ,$$

which contradicts the assumption $\mathfrak{p} \in \mathrm{Ass}_R(D)$.
(iii) The first statement implies that there is an associated prime of D which contains f and (ii) shows that every such prime ideal $\mathfrak{p}$ has height smaller than h.

4.3 The case $h = \dim(R) - 2$, i. e. the set $\mathrm{Ass}_R(D(\mathrm{H}^{n-2}_{(X_1,\ldots,X_{n-2})R}(k[[X_1, \ldots, X_n]])))$

We can give a partial answer to the question which height one prime ideals contained in I are associated to D:

4.3.1 Corollary

If we are in the special case where $h = n - 2$, R_0 is regular and $\mathrm{height}(I_0) < h$ in the situation of theorem 4.2.3, we clearly have (because of theorem 4.2.3 (ii))

$$fR \in \mathrm{Ass}_R(D) \iff \mathrm{H}^{n-2}_{I_0}(R_0) \neq 0 \quad .$$

In this case, fR is a maximal element of $\mathrm{Ass}_R(D)$. By [HL, Theorem 2.9], the latter holds if and only if $\dim(R_0/I_0) \geq 2$ and $\mathrm{Spec}(\overline{R_0}/I_0\overline{R_0}) \setminus \{\mathfrak{m}_0(\overline{R_0}/I_0\overline{R_0})\}$ is connected, where $\overline{R_0}$ is defined as the completion of the strict henselization of R_0; this means that $\overline{R_0}$ is obtained from R_0 by replacing the coefficient field k by its separable closure in any fixed algebraic closure of k.

4.3.2 Remarks

(i) In the situation of the statement (i) of theorem 4.2.3, it can easily happen that both $f \in I$ and $\mathrm{H}^h_{I_0}(R_0) = 0$ hold; then we have, in particular, $fR \notin \mathrm{Ass}_R(D)$. Hence, in general, not all height one prime ideals contained in I are associated to D. In fact, if $\mathrm{height}(I_0) < h$, then f is necessarily contained in I. Hence, if $\mathrm{ara}(I_0) < h$, then both $f \in I$ and $\mathrm{H}^h_{I_0}(R_0) = 0$ hold and therefore one has $fR \notin \mathrm{Ass}_R(D)$.
(ii) In the situation of corollary 4.3.1, it can happen that $fR \in \mathrm{Ass}_R(D)$. For example, we can take

$$n = 5, h = 3, k = \mathbf{Q} \text{ (the rationals)}, \; R_0 = \mathbf{Q}[[y_1, y_2, y_3, y_4]] \quad ,$$

a power series algebra over $\mathbf{Q}$ in the variables y_1, y_2, y_3, y_4. We set

$$x_1 = y_1y_3, x_2 = y_2y_4, x_3 = y_1y_4 + y_2y_3, x_4 = y_1 + y_3, x_5 = y_2 + y_4 \quad .$$

Then $\mathrm{height}(I_0) = 2$ and $\mathrm{H}^3_{I_0}(R_0) \neq 0$. Furthermore

$$f := -X_2X_4^2 + X_3X_4X_5 - X_1X_5^2 + 4X_1X_2 - X_3^2 \in R$$

generates the kernel of the $\mathbf{Q}$-algebra homomorphism $R \to R_0$, which is determined by $X_i \mapsto x_i$ $(i = 1, \ldots, 5)$, where R is defined as the power series ring $\mathbf{Q}[[X_1, X_2, X_3, X_4, X_5]]$ over $\mathbf{Q}$ in the variables X_1, X_2, X_3, X_4, X_5. Now, by corollary 4.3.1, fR is a maximal element of $\mathrm{Ass}_R(D)$. In particular, this example clearly provides a counterexample to conjecture (+) from [HS1, section 0] (see also remark 1.2.4 for details on this).
Proof:
(i) We assume that $\mathrm{height}(I_0) < h$ and prove $f \in I$: Let $\mathfrak{p}_0$ be a prime ideal minimal over I_0 and such that $\mathrm{height}(\mathfrak{p}_0) < h - 1$; the inclusion $\mathfrak{p}_0 \cap R \supseteq I + fR$ induces a surjection $R/(I + fR) \to R/\mathfrak{p}_0 \cap R$; on the other hand, $R_0/\mathfrak{p}_0$ is finite over $R/\mathfrak{p}_0 \cap R$. Therefore we have

$$\dim(R_0/\mathfrak{p}_0) = \dim(R/\mathfrak{p}_0 \cap R) \leq \dim(R/(I + fR)) \quad .$$

Now, if f was not contained in I, one would conclude $\dim(R_0/\mathfrak{p}_0) \leq n - h - 1$ and, hence, $\mathrm{height}(\mathfrak{p}_0) \geq h$.
(ii) It is easy to see that

$$\sqrt{I_0} = (y_1, y_2)R_0 \cap (y_3, y_4)R_0$$

and so $\mathrm{height}(I_0) = 2$ and a Mayer-Vietoris sequence argument, applied to the ideals $(y_1, y_2)R_0$ and $(y_3, y_4)R_0$, shows that $\mathrm{H}^3_{I_0}(R_0) \neq 0$. f generates the kernel of the $\mathbf{Q}$-algebra homomorphism $R \to R_0$; this can be seen e. g. by observing that f, as an element of $\mathbf{Q}[X_1, X_2, X_3, X_4, X_5]$, generates the kernel of the associated map over polynomial instead of formal power series rings, which in turn is true, because first of all an easy calculation shows that f is in this kernel and, secondly, as a polynomial, f is irreducible, which can either be seen by a direct calculation or by using a computer algebra system like, for example, Macaulay 2. The rest follows from corollary 4.3.1.

Assume that $p \in I$ is a prime element. The next example and, more generally, theorem 4.3.4 show that under certain conditions, p is contained in infinitely many associated height two prime ideals of D. This is useful for two reasons: It will lead to a generalization of an example of Hartshorne of a non-artinian (but zero-dimensional) local cohomology module (see theorem 6.2.3); and secondly, it will show that either conjecture (*) holds (for $h = n-2$) or, if not, D satisfies a remarkable property (see remark 4.3.6 for details on this property).

4.3.3 Example
Still in the above situation consider $p := X_1X_n + X_2X_{n-1} \in I \cap (X_{n-1}, X_n)R$. For every $\lambda \in k$ set $\mathfrak{p}_\lambda := (X_{n-1} + \lambda X_1, X_n - \lambda X_2)R$. Then $\mathfrak{p}_\lambda \in \mathrm{Ass}_R(D(\mathrm{H}_I^{n-2}(R)))$ holds (this follows from theorem 3.1.3 (v)) and

$$p = X_1(X_n - \lambda X_2) + X_2(X_{n-1} + \lambda X_1)$$

is contained in every $\mathfrak{p}_\lambda$.

The same idea works more general:

4.3.4 Theorem
Let $R = k[[X_1, \ldots, X_n]]$ be a power series ring in the variables $X_1, \ldots, X_n$ ($n \geq 4$) over a field k and let I be the ideal $(X_1, \ldots, X_{n-2})R$ (i. e. $h = n-2$ in the above notation). Furthermore, let $p \in R$ be a prime element such that $p \in I \cap (X_{n-1}, X_n)R$.
The set

$$\{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}_I^{n-2}(R))), p \in \mathfrak{p}, \mathrm{height}(\mathfrak{p}) = 2\}$$

is infinite.
Proof:
It is easy to see that there exist $f, g \in I, f \notin (X_{n-1}, X_n)R$ and $l \geq 1$ such that

$$p = X_n^l f + X_{n-1} g$$

holds. Let $m \in \mathbb{N}^+$ be arbitrary. We have

$$p = (X_n^l + X_1^m g)f + (X_{n-1} - X_1^m f)g$$

and so

$$p \in I_m := (X_n^l + X_1^m g, X_{n-1} - X_1^m f)R \quad .$$

The elements

$$X_1, \ldots, X_{n-2}, X_n^l + X_1^m g, X_{n-1} - X_1^m f$$

form a system of parameters of R and so there exists a $\mathfrak{p}_m \in \mathrm{Ass}_R(D(\mathrm{H}_I^{n-2}(R)))$ containing I_m. For $m, m' \in \mathbb{N}^+, m \neq m'$

$$\sqrt{I_m + I_{m'}} = (X_1, X_n, X_{n-1})R \cap \sqrt{(X_{n-1}, X_n, f, g)R}$$

holds; in particular, all primes containing $I_m + I_{m'}$ have height at least three. The statement follows.

4.3.5 Remark
In the situation of theorem 4.3.4, conjecture (*) would clearly imply $pR \in \mathrm{Ass}_R(D)$. Now, if pR was not associated to D, there would be a remarkable consequence, which is somewhat counterintuitive (note that,

in the situation below, the way we choose $\mathfrak{p}_l$ has nothing to do with the way how we choose $d_{l+1}, d_{l+2}, \ldots$) and which is explained in the next remark.

4.3.6 Remark

Let $R = k[[X_1, \ldots, X_n]]$ be a power series algebra over a field k in the variables $X_1, \ldots, X_n$; set $I = (X_1, \ldots, X_{n-2})R, D := D(\mathrm{H}_I^{n-2}(R))$ and $Y := X_1 \cdot \ldots \cdot X_{n-2}$; furthermore, assume that $p \in I$ is a prime element of R such that there are infinitely many height two prime ideals associated to D and containing p (by theorem 4.3.4, this is true for example, if $p \in (X_{n-1}, X_n)R$ holds) and such that $pR \notin \mathrm{Ass}_R(D)$.

Then for any sequence $(\mathfrak{p}_i)_{i \in \mathrm{N}}$ of pairwise different elements of $\mathrm{Ass}_R(\mathrm{Hom}_R(R/pR, D))$ and for any sequence $(d_i)_{i \in \mathrm{N}}$ in D such that $\mathrm{Ann}_R(d_i) = \mathfrak{p}_i$ for every $i \in \mathrm{N}$, there exists a number N such that

$$\mathrm{Ann}_R(d_{l+1}Y^{l+1} + d_{l+2}Y^{l+2} + \ldots) \subseteq \mathfrak{p}_l$$

holds for every $l > N$ (see the proof below for remarks on our notation).

Proof:

It is well-known that $\mathrm{H}_I^{n-2}(R)$ is the cohomology in the $n-2$-th degree of the Čech-complex

$$0 \to R \to \oplus_{i_1=1}^{n-2} R_{X_{i_1}} \to \oplus_{1 \leq i_1 < i_2 \leq n-2} R_{X_{i_1} X_{i_2}} \to \ldots \to R_{X_1 \ldots X_{n-2}} \to 0 \quad ;$$

Therefore we can write $\mathrm{H}_I^{n-2}(R)$ as

$$k[[X_{n-1}X_n]][X_1^{-1}, \ldots, X_{n-2}^{-1}] \quad ;$$

by definition, this expression shall stand for

$$\oplus_{i_1, \ldots, i_{n-2} \leq 0} k[[X_{n-1}, X_n]] \cdot X_1^{i_1} \cdot \ldots \cdot X_{n-2}^{i_{n-2}}$$

with the obvious R-module structure on it. Using this, a straight-forward calculation shows

$$D = k[X_{n-1}^{-1}, X_n^{-1}][[X_1, \ldots, X_{n-2}]] \quad ,$$

where we use similar notation like above, i. e. we write the elements of D as formal power series in $X_1, \ldots, X_{n-2}$ and coefficients in

$$k[X_{n-1}^{-1}, X_n^{-1}] = \oplus_{i_{n-1}, i_n \leq 0} k \cdot X_{n-1}^{i_{n-1}} \cdot X_n^{i_n} \quad .$$

Using this description of D it is clear that $d_{l+1}Y^{l+1} + d_{l+2}Y^{l+2} + \ldots$ is an element of D for every $l \in \mathrm{N}$. In the same way it is clear that the element

$$d := d_0 + Y \cdot d_1 + Y^2 \cdot d_2 + \ldots \in D$$

is well-defined. By construction p annihilates d and so, because of $pR \notin \mathrm{Ass}_R(D)$, there exists $r \in \mathrm{Ann}_R(d) \setminus pR$. We conclude

$$0 = rd = rd_0 + rYd_1 + rY^2 d_2 + \ldots$$

The height of every prime ideal associated to D is at most two and thus for every $l \in \mathrm{N}$ either $\mathrm{Ann}_R(rd_l) = \mathfrak{p}_l$ or $rd_l = 0$ holds. The latter condition is equivalent to $r \in \mathfrak{p}_l$, which holds if and only if $\mathfrak{p}_l$ contains the ideal $(r, p)R$. Hence there are only finitely many $l \in \mathrm{N}$ such that $rd_l = 0$, the set

$$M := \{l \in \mathrm{N} | rd_l = 0\}$$

is finite. For every $l \in \mathbb{N} \setminus M$ we have

$$\mathrm{Ann}_R(d_{l+1}Y^{l+1} + d_{l+2}Y^{l+2} + \ldots) \subseteq \mathrm{Ann}_R(rd_{l+1}Y^{l+2} + rd_{l+2}Y^{l+2} + \ldots) = \mathrm{Ann}_R(rd_0 + \ldots + rd_lY^l) \subseteq \mathfrak{p}_l \quad ;$$

note that the last inclusion follows from lemma 4.3.7 below. In particular, for every $l > \max M$ we have

$$\mathrm{Ann}_R(d_{l+1}Y^{l+1} + d_{l+2}Y^{l+2} + \ldots) \subseteq \mathfrak{p}_l \quad ,$$

we can take $N := \max M$.

4.3.7 Lemma
In the situation of theorem 4.3.4, assume that $d_1, \ldots, d_n$ are elements of D such that for every $i = 1, \ldots, n$ the ideal $\mathrm{Ann}_R(d_i) =: \mathfrak{p}_i$ is a height-two prime ideal of D and such that the $\mathfrak{p}_i$ are pairwise different. Then

$$\mathrm{Ann}_R(d_1 + \ldots + d_n) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n$$

holds.
Proof:
By induction on n, the case $n = 1$ being trivial. We assume that $n > 1$ and that the lemma is true for smaller n. The inclusion $\supseteq$ is trivial. Now, if there was an element $r \in \mathrm{Ann}_R(d_1 + \ldots + d_n) \setminus \mathfrak{p}_i$ for some $i \in \{1, \ldots, n\}$, we would have

$$-rd_1 = rd_2 + \ldots + rd_n$$

and

$$\mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_n = \mathrm{Ann}_R(d_2 + \ldots + d_n) \subseteq \mathrm{Ann}_R(rd_2 + \ldots + rd_n) = \mathfrak{p}_1 \quad ,$$

which would be a contradiction.

5 On the meaning of a small arithmetic rank of a given ideal

We investigate the condition that the arithmetic rank of a given ideal is small in the sense that it is one or two. We start with an example of an ideal whose cohomological dimension is one but whose arithmetic rank is two (example 5.1); this makes the question when $\operatorname{ara}(I) \leq 1$ holds more difficult; we present criteria for this condition and also for $\operatorname{ara}(I) \leq 2$ (theorem 5.2.5 and corollary 5.2.6). While this works equally well in the local and in the graded case, we distinguish some subtle differences between these two cases in the third subsection 5.3.

5.1 An Example

We start with an example of an ideal I of a noetherian ring R where both $0 = \mathrm{H}_I^2(R) = \mathrm{H}_I^3(R) = \ldots$ and $\operatorname{ara}(I) \geq 2$ hold: Let k be any field and $R = k[[x, y, z, w]]$ a power series ring over k in four variables. Set

$$f = xw - yz \quad ,$$

$$g_1 = y^3 - x^2 z, g_2 = z^3 - w^2 y$$

and

$$I = \sqrt{(f, g_1, g_2)R} \quad .$$

The ideal $I \subseteq R$ is the complete version of the vanishing ideal of a rational quartic curve in projective three-space over k; it is well-known that $I \subseteq R$ is a height-two prime ideal of R. We claim that both $\mathrm{H}_I^s(R/fR) = 0$ for every $s \geq 2$ and $\operatorname{ara}(I(R/fR)) \geq 2$ hold (the last statement may be known, we include a proof for lack of a reference):

Let $y_0, \ldots, y_3$ be new variables and set $S := k[[y_0, y_1, y_2, y_3]]$. Denote by R_1 the three-dimensional subring $R_1 := k[[y_0 y_1, y_0 y_2, y_1 y_3, y_2 y_3]]$ of S. The ring homomorphism

$$R \to R_1, x \mapsto y_0 y_1, y \mapsto y_0 y_2, z \mapsto y_1 y_3, w \mapsto y_2 y_3$$

clearly induces an isomorphism

$$R/fR \cong R_1 (\subseteq S) \quad .$$

Now consider the k-linear map

$$k[y_0, y_1, y_2, y_3] \xrightarrow{\varphi} R_1$$

that sends a term $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ to $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3} \in R_1$ if $\alpha_0 + \alpha_3 = \alpha_1 + \alpha_2$ holds, and to zero otherwise. Note that φ is well-defined by construction and naturally induces a map

$$S = k[[y_0, y_1, y_2, y_3]] \xrightarrow{\tilde{\varphi}} R_1 \quad .$$

Now it is easy to see that $\tilde{\varphi}$ is R_1-linear and makes R_1 into a direct summand in S (as an R_1-submodule). Thus $\mathrm{H}_I^2(R/fR)$ is isomorphic to a direct summand of $\mathrm{H}_{IS}^2(S)$. We have

$$IS = (g_1, g_2)S = ((y_0 y_2^3 - y_1^3 y_3) \cdot y_0^2, (y_0 y_2^3 - y_1^3 y_3) \cdot (-y_3^2))S$$

and

$$\sqrt{IS} = (y_0 y_2^3 - y_1^3 y_3)S \quad .$$

This implies $\mathrm{H}^2_{IS}(S) = 0$ and thus, by what we have seen above, $\mathrm{H}^2_I(R/fR) = 0$.

Now we show $\operatorname{ara}(I(R/fR)) = 2$: We assume $\operatorname{ara}(I(R/fR)) \neq 2$; then we clearly have $\operatorname{ara}(I(R/fR)) = 1$. Let $h \in R$ be such that

$$I(R/fR) = \sqrt{h(R/fR)}$$

holds. This implies

$$\sqrt{IS} = \sqrt{hS} \quad .$$

We have seen before that

$$\sqrt{IS} = (y_0 y_2^3 - y_1^3 y_3)S$$

holds. S is a unique factorization domain and so there exist $N \geq 1$ and $s \in S$ such that

$$h = (y_0 y_2^3 - y_1^3 y_3)^N \cdot s \text{ and } (y_0 y_2^3 - y_1^3 y_3) \not| s$$

hold. From $h \in R_1 \subseteq S$ it follows that all terms $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ in $h \in S$ have the property $\alpha_0 + \alpha_3 = \alpha_1 + \alpha_2$; on the other hand, all terms $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ of $(y_0 y_2^3 - y_1^3 y_3)^N$ have the property $(\alpha_0 + \alpha_3) - (\alpha_1 + \alpha_2) = -2N$. So we can assume that all terms $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}$ of s have the property $(\alpha_0 + \alpha_3) - (\alpha_1 + \alpha_2) = 2N$. But then s cannot be a unit in S and so

$$(y_0 y_2^3 - y_1^3 y_3)S = \sqrt{hS} = (y_0 y_2^3 - y_1^3 y_3)S \cap \sqrt{sS}$$

clearly leads to a contradiction.

5.2 Criteria for $\operatorname{ara}(I) \leq 1$ and $\operatorname{ara}(I) \leq 2$

5.2.1 Remark

Let $(R, \mathfrak{m})$ be a noetherian local ring. By $\mathrm{E} := \mathrm{E}_R(R/\mathfrak{m})$ we denote an R-injective hull of $R/\mathfrak{m}$. Let I be an ideal of R. Then the following statements are equivalent:

(i) $\operatorname{ara}(I) \leq 1$.
(ii) $\mathrm{H}^i_I(R) = 0$ for $i \geq 2$ and $\exists f \in I : f$ operates surjectively on $\mathrm{H}^1_I(R)$.
(iii) $\mathrm{H}^i_I(R) = 0$ for $i \geq 2$ and $\exists f \in I : f$ operates injectively on $D(\mathrm{H}^1_I(R))$.
(iv) $\mathrm{H}^i_I(R) = 0$ for $i \geq 2$ and $I \not\subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(D(\mathrm{H}^1_I(R)))} \mathfrak{p}$.

Furthermore, if conditions (ii) or (iii) hold, we have $\sqrt{I} = \sqrt{fR}$.
Proof:
(ii) – (iv) are obviously equivalent, we show (i) $\iff$ (ii):
(i) $\Rightarrow$ (ii): Assume $\sqrt{I} = \sqrt{fR}$ for some $f \in R$. f clearly operates surjectively on $\mathrm{H}^1_{fR}(R)$; but $\sqrt{I} = \sqrt{fR}$ implies $\mathrm{H}^1_I = \mathrm{H}^1_{fR}$.
(ii) $\Rightarrow$(i): $\mathrm{H}^1_I(\_)$ is right-exact on the category of R-modules. Therefore we have an exact sequence

$$\mathrm{H}^1_I(R) \xrightarrow{f} \mathrm{H}^1_I(R) \to \mathrm{H}^1_I(R/fR) \to 0 \quad .$$

Thus $\mathrm{H}^1_I(R/fR) = 0$ holds, implying $\mathrm{H}^1_I(R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ containing f. But because of our hypothesis, this means that we have $\mathrm{H}^i_I(R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ containing I and for all $i \geq 1$. Thus f must be contained in every prime ideal of R containing I. $\sqrt{I} = \sqrt{fR}$ follows.

5.2.2 Remark

Now we consider the following situation (referred to from now on as graded situation): Let K be a field, $l \in \mathbb{N}$,

$$R = K[X_0, \ldots, X_N]/J$$

($N \in \mathbb{N}, J \subseteq K[X_0, \ldots, X_N]$ a homogenous ideal, where every X_i has a multidegree in $\mathbb{N}^l$),

$$I \subseteq R \text{ a homogenous ideal,}$$

$\mathfrak{m}$ the maximal homogenous ideal $(X_0, \ldots, X_N)R$ of R, $\mathrm{E} := \mathrm{E}_R(R/\mathfrak{m})$ an R-injective hull of $R/\mathfrak{m}$; E has a natural grading and serves also as a *-R-injective hull of $R/\mathfrak{m}$. Here we follow the use of *-notation from [BS, in particular sections 12 and 13]. $*D$ shall denote the functor from the category of graded R-modules to itself defined by

$$(*D)(M) := *\mathrm{Hom}_R(M, \mathrm{E})$$

for a graded R-module M. We define the homogenous arithmetic rank of I to be

$$\mathrm{ara}^h(I) := \min\{l \in \mathbb{N} | \exists r_1, \ldots, r_l \in R \text{ homogenous} : \sqrt{I} = \sqrt{(r_1, \ldots, r_l)R}\}$$

and we set

$$I^h := \{x \in I | x \text{ is homogenous}\} \quad .$$

Now, just like in the local case, one can show that the following statements are equivalent:

(i) $\mathrm{ara}^h(I) \leq 1$.

(ii) $\mathrm{H}^i_I(R) = 0$ for $i \geq 2$ and $\exists$ homogenous $f \in I : f$ operates surjectively on $\mathrm{H}^1_I(R)$.

(iii) $\mathrm{H}^i_I(R) = 0$ for $i \geq 2$ and $\exists$ homogenous $f \in I : f$ operates injectively on $(*D)(\mathrm{H}^1_I(R))$.

(iv) $\mathrm{H}^i_I(R) = 0$ for $i \geq 2$ and $I^h \not\subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R((*D)(\mathrm{H}^1_I(R)))} \mathfrak{p}$.

Furthermore, if conditions (ii) or (iii) hold, we have $\sqrt{I} = \sqrt{fR}$.

5.2.3 Definition

Let $(R, \mathfrak{m})$ be a noetherian local ring and $X \subseteq \mathrm{Spec}(R)$ a subset. We say that X satisfies prime avoidance if, for every ideal J of R,

$$J \subseteq \bigcup_{\mathfrak{p} \in X} \mathfrak{p}$$

implies

$$\exists \mathfrak{p}_0 \in X : J \subseteq \mathfrak{p}_0 \quad .$$

5.2.4 Definition

In the graded situation, let $X \subseteq \mathrm{Spec}^h(R) := \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p} \text{ homogenous}\}$ be any subset. We say that X satisfies homogenous prime avoidance if, for every homogenous ideal J of R,

$$J^h \subseteq \bigcup_{\mathfrak{p} \in X} \mathfrak{p}$$

implies

$$\exists \mathfrak{p}_0 \in X : J \subseteq \mathfrak{p}_0 \quad .$$

5.2.5 Theorem

(i) Let $(R, \mathfrak{m})$ be a noetherian local ring and I an ideal of R such that

$$0 = \mathrm{H}^2_I(R) = \mathrm{H}^3_I(R) = \ldots \tag{1}$$

holds. Then

$$\mathrm{ara}(I) \leq 1 \iff \mathrm{Ass}_R(D(\mathrm{H}^1_I(R))) \text{ satisfies prime avoidance} \quad .$$

(ii) Let R be graded and $I \subseteq R$ an homogenous ideal such that $0 = \mathrm{H}^2_I(R) = \mathrm{H}^3_I(R) = \ldots$. Then

$$\mathrm{ara}^h(I) \leq 1 \iff \mathrm{Ass}_R((*D)(\mathrm{H}^1_I(R))) \text{ satisfies homgenous prime avoidance} \quad .$$

Proof:

(i) We set

$$D := D(\mathrm{H}^1_I(R)) \quad .$$

$\Rightarrow$: Let $J \subseteq R$ be an ideal such that

$$J \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(D)} \mathfrak{p} \quad .$$

We claim that $\mathrm{Hom}_R(R/J, D) \neq 0$. Assumption: $\mathrm{Hom}_R(R/J, D) = 0$: It is a general fact that for every ideal $K \subseteq R$ and every R-module M one has

$$\mathrm{Hom}_R(R/K, D(M)) = D(M/KM) \quad .$$

(Proof of this general fact: If $K = (k_1, \ldots k_l)R$ for some $k_1, \ldots, k_l \in R$, the exact sequence

$$R^l \stackrel{(k_1,\ldots,k_l)}{\to} R \stackrel{\text{can.}}{\to} R/K \to 0$$

induces an exact sequence

$$M^l \stackrel{(k_1,\ldots,k_l)}{\to} M \stackrel{\text{can.}}{\to} M/KM \to 0 \quad ;$$

The functor D is exact and so we get an exact sequence

$$0 \to D(M/KM) \stackrel{\text{can.}}{\to} D(M) \stackrel{\begin{pmatrix} k_1 \\ \vdots \\ k_l \end{pmatrix}}{\to} D(M)^l \quad ,$$

from which the statement $\mathrm{Hom}_R(R/K, D(M)) = D(M/KM)$ follows.)

We apply this general fact in the case $K = J$, $M = \mathrm{H}^1_I(R)$ and conclude that

$$0 = \mathrm{Hom}_R(R/J, D(\mathrm{H}^1_I(R))) = D(\mathrm{H}^1_I(R)/J\,\mathrm{H}^1_I(R)) = D(\mathrm{H}^1_I(R/J)).$$

For the last equality we use the fact that the functor H^1_I is right-exact (because of hypothesis (1): $0 = \mathrm{H}^2_I(R) = \mathrm{H}^3_I(R) = \ldots$). But $D(\mathrm{H}^1_I(R/J)) = 0$ implies that $\mathrm{H}^1_I(R/J) = 0$. Again, because of hypothesis (1), it follows that $\mathrm{H}^1_I(R/\mathfrak{p}) = 0$ for all prime ideals $\mathfrak{p}$ of R containing J. Clearly, the last condition implies $I \subseteq \mathfrak{p}$

for all $\mathfrak{p}$ containing J, that is $I \subseteq \sqrt{J}$. There is an $x \in R$ such that $\sqrt{I} = \sqrt{xR}$. Hence $x^l \in J$ for $l >> 0$. So there is a $\mathfrak{p} \in \mathrm{Ass}_R(D)$ containing x. Now we have

$$0 = \mathrm{H}^1_{xR}(R/\mathfrak{p}) = \mathrm{H}^1_I(R/\mathfrak{p})$$

and thus

$$0 = D(\mathrm{H}^1_I(R/\mathfrak{p})) = \mathrm{Hom}_R(R/\mathfrak{p}, D(\mathrm{H}^1_I(R)))$$

contradicting $\mathfrak{p} \in \mathrm{Ass}_R(D)$. Thus the assumption $\mathrm{Hom}_R(R/J, D) = 0$ is false and the claim $\mathrm{Hom}_R(R/J, D) \neq 0$ is proven; so there exists a $d \in D \setminus \{0\}$ such that $J \subseteq \mathrm{ann}_R(d)$.
$\Leftarrow$: We have to show the existence of an $x \in I$ operating surjectively on $\mathrm{H}^1_I(R)$. Assume to the contrary

$$I \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(D)} \mathfrak{p} \quad .$$

From the hypothesis we get a $\mathfrak{p}_0 \in \mathrm{Ass}_R(D)$ such that $I \subseteq \mathfrak{p}_0$. But this $\mathfrak{p}_0$ would satisfy

$$0 \neq \mathrm{H}^1_I(R/\mathfrak{p}_0) = 0 \quad .$$

(ii) The proof consists mainly of a graded version of the proof of (i):
$\Rightarrow$: Let $J \subseteq R$ be an homogenous ideal such that

$$J^h \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R((*D)(\mathrm{H}^1_I(R)))} \mathfrak{p}$$

and $x \in R^h$ an element such that $\sqrt{I} = \sqrt{xR}$. We assume

$$\mathrm{Hom}_R(R/J, *\mathrm{Hom}_R(\mathrm{H}^1_I(R), \mathrm{E})) = 0$$

and remark that for the first Hom (in the preceding formula) it would not make any difference if we replaced Hom by $*\,$Hom. This implies

$$*\mathrm{Hom}_R((R/J) \otimes_R \mathrm{H}^1_I(R), \mathrm{E}) = 0$$

and hence $\mathrm{H}^1_I(R/J) = 0$. Thus $I \subseteq \mathfrak{q}$ for all prime ideals $\mathfrak{q}$ of R containing J. This implies the existence of a $\mathfrak{p}_0 \in \mathrm{Ass}_R((*D)(\mathrm{H}^1_I(R)))$ such that $x \in \mathfrak{p}_0$ contradicting $\mathrm{H}^1_I(R/\mathfrak{p}_0) \neq 0$.
$\Leftarrow$: We assume that for every $x \in I^h$ there exists a $\mathfrak{p} \in \mathrm{Ass}_R((*D)(\mathrm{H}^1_I(R)))$ such that $x \in \mathfrak{p}$, i. e.

$$I^h \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(*\mathrm{Hom}_R(\mathrm{H}^1_I(R), \mathrm{E}))} \mathfrak{p} \quad .$$

There is a $\mathfrak{p}_0 \in \mathrm{Ass}_R(*\mathrm{Hom}_R(\mathrm{H}^1_I(R), \mathrm{E}))$ containing I, contradicting $\mathrm{H}^1_I(R/\mathfrak{p}_0) \neq 0$.

Theorem 5.2.5 implies criteria for $\mathrm{ara}(I) \leq 2$ resp. for $\mathrm{ara}^h(I) \leq 2$:

5.2.6 Corollary

(i) Let $(R, \mathfrak{m})$ be a noetherian local ring and I an ideal of R. Then $\mathrm{ara}(I) \leq 2$ if and only if there exists $g \in I$ such that $0 = \mathrm{H}^2_I(R/gR) = \mathrm{H}^3_I(R/gR) = \ldots$ and such that $\mathrm{Ass}_R(D(\mathrm{H}^1_I(R/gR))$ satisfies prime avoidance.

(ii) Let R be a graded ring and I an ideal of R. Then $\mathrm{ara}^h(I) \le 2$ if and only if there exists a homogenous $g \in I$ such that $0 = \mathrm{H}^2_I(R/gR) = \mathrm{H}^3_I(R/gR) = \ldots$ and such that $\mathrm{Ass}_R(D(\mathrm{H}^1_I(R/gR))$ satisfies homogenous prime avoidance.
Proof:
$\Rightarrow$ follows immediately from theorem 5.2.5 (i) resp. (ii); for the other implication observe that the conditions on the right side imply $\mathrm{ara}_{R/gR}(I/(gR)) = 1$ resp. $\mathrm{ara}^h_{R/gR}(I/(gR)) = 1$ again by theorem 5.2.5 (i) resp. (ii).

5.3 Differences between the local and the graded case

5.3.1 Lemma

Let R be a graded domain and $f \in R\backslash\{0\}$. Then the ideal $\sqrt{fR}$ is homogenous if and only if f is homogenous. In particular, for any homogenous ideal I of R we have

$$\mathrm{ara}(I) \le 1 \iff \mathrm{ara}^h(I) \le 1 \quad .$$

Proof:
$\Leftarrow$ is clear. $\Rightarrow$: R is $\mathbb{N}^l$-graded. This given grading may be seen as l given $\mathbb{N}$-gradings on R and so we may assume $l = 1$. Let $\delta := \deg(f)$. Then f_δ(=degree-δ-part of f) $\in \sqrt{fR}$, i. e. $\exists n \in \mathbb{N}^+$ and $\exists g \in R : f_\delta^n = fg$. R is a domain and so f (as well as g) must be homogenous.

5.3.2 Remark

In the graded situation, given graded R-modules M and N,

$$*\,\mathrm{Hom}_R(M,N) \subseteq \mathrm{Hom}_R(M,N)$$

holds. For finite M one has equality here, but for arbitrary M equality does not hold in general. In fact one has

$$\mathrm{Ass}_R(*\,\mathrm{Hom}_R(M,N)) \subsetneq \mathrm{Ass}_R(\mathrm{Hom}_R(M,N))$$

in general as we will see below in the case $M = \mathrm{H}^1_I(R)$, $N = \mathrm{E} := \mathrm{E}_R(R/\mathfrak{m})$; then we will also see that, in some sense, $\mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{H}^1_I(R),\mathrm{E}))$ is much larger than $\mathrm{Ass}_R(*\,\mathrm{Hom}_R(\mathrm{H}^1_I(R),\mathrm{E}))$.

5.3.3 Definition and remark

Let R be a graded ring and $I \subseteq R$ homogenous ideal such that $0 = \mathrm{H}^2_I(R) = \mathrm{H}^3_I(R) = \ldots$. Let $f \in I$ be an element, not necessarily homogenous. Now we define two conditions on I and f:

(C_1) $$\forall_{\mathfrak{p}\in \mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{H}^1_I(R),\mathrm{E}))} f \notin \mathfrak{p}$$

(C_2) $$\forall_{\mathfrak{p}\in \mathrm{Ass}_R(*\,\mathrm{Hom}_R(\mathrm{H}^1_I(R),\mathrm{E}))} f \notin \mathfrak{p}$$

Condition (C_1) is just a reformulation of $\sqrt{(I)} = \sqrt{fR}$ (see the proof of remark 5.2.1). In contrary to (C_1), all objects in (C_2) are graded and so condition (C_2) may be seen as a graded version of the condition "f generates I up to radical"; furthermore (C_1) clearly implies (C_2).

Terminology: For a given homogenous ideal I of R and a given element $f \in I$ we say that condition $(C_i)(I; f)$ holds if (C_i) holds for I and f $(i = 1, 2)$.

In the next section we will investigate to what extent condition (C_1) differs from condition (C_2). Theorem 5.3.5 will show that there are (in fact many) non-homogenous $f \in I$ such that $(C_2)(I; f)$ holds, but there are no non-homogenous $f \in I$ such that $(C_1)(I; f)$ holds.

5.3.4 Remark

It is easy to see that for every homogenous element $g \in I$ the conditions $(C_1)(I; f)$ and $(C_2)(I; f)$ are equivalent.

5.3.5 Theorem

(i) Let I be a homogenous ideal of a graded ring R such that $\mathrm{ara}^h(I) \leq 1$. Let $g_1, \ldots, g_n \in I \setminus \{0\}$ be homogenous of pairwise different degrees (in $\mathbb{N}^l$) and such that

$$\sqrt{I} = \sqrt{(g_1, \ldots, g_n)R} \quad .$$

Then

$$(C_2)(I; g_1 + \ldots + g_n) \text{ holds.}$$

(ii) Let I be a homogenous ideal of a graded ring R (and such that $0 = \mathrm{H}^2_I(R) = \mathrm{H}^3_I(R) = \ldots$). Let $g \in I$ be a non-homogenous element. Then

$$(C_1)(I; g) \text{ does not hold.}$$

Proof:

(i) We have $\mathrm{H}^1_I(R/(g_1, \ldots, g_n)R) = 0$ and hence

$$(g_1, \ldots, g_n)R \not\subseteq \mathfrak{p}$$

for all $\mathfrak{p} \in \mathrm{Ass}_R(* \mathrm{Hom}_R(\mathrm{H}^1_I(R), \mathrm{E}))$. Theorem 5.2.5 (ii) implies

$$((g_1, \ldots, g_n)R)^h \not\subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(* \mathrm{Hom}_R(\mathrm{H}^1_I(R), \mathrm{E}))} \mathfrak{p} \quad .$$

Because of the different degrees of the g_i we conclude

$$(g_1 + \ldots + g_n)R \not\subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(* \mathrm{Hom}_R(\mathrm{H}^1_I(R), \mathrm{E}))} \mathfrak{p}$$

and the statement follows.

(ii) The first statement of lemma 5.3.1 implies that if R is a graded domain and $I \subseteq R$ is a homogenous ideal such that $\mathrm{ara}(I) \leq 1$ ($\iff$ $\mathrm{ara}^h(I) \leq 1$), every non-homogenous $g \in I$ does not operate injectively on $\mathrm{Hom}_R(\mathrm{H}^1_I(R), \mathrm{E})$. Furthermore, if $\mathrm{ara}(I) > 1$ ($\iff$ $\mathrm{ara}^h(I) > 1$), it is clear (use the ideas of section 5.2) that no $g \in I$ operates injectively on $\mathrm{Hom}_R(\mathrm{H}^1_I(R), \mathrm{E})$.

5.3.6 Remark

While in the situation of theorem 5.3.5 statement (i) says there are (many) non-homogenous $f \in I$ operating injectively on $(*D)(\mathrm{H}^1_I(R))$, (ii) says there are no non-homogenous $f \in I$ operating injectively on $D(\mathrm{H}^1_I(R))$.

6 Applications

6.1 Hartshorne-Lichtenbaum vanishing

The (more difficult) part of Hartshorne-Lichtenbaum vanishing theorem (for another reference, see, e. g., [BS, 8.2.1]) says that for an ideal I in a noetherian local complete domain $(R, \mathfrak{m})$ there is the implication

$$\mathrm{H}_I^{\dim(R)}(R) \neq 0 \Longrightarrow \sqrt{I} = \mathfrak{m} \quad .$$

We present two new proofs for it: theorem 6.1.2 works with the normalization of R and the Matlis dual of the local cohomology module in question, while theorem 6.1.4 uses the fact that, over a noetherian local complete Gorenstein ring $(S, \mathfrak{m})$ of dimension $n+1$ and every height n prime ideal $\mathfrak{P}$ in S, one has $D(\mathrm{H}^n_{\mathfrak{P}}(S)) = \widehat{S_{\mathfrak{P}}}/S$ (this is lemma 3.2.1); it is remarkable that the proof of theorem 6.1.4 uses (this is hidden in the proof of lemma 6.1.3) the ring structure on $\widehat{S_{\mathfrak{P}}}$.

6.1.1 Theorem

Let $(R, \mathfrak{m})$ be a noetherian local ring and M a finitely generated R-module. Then $\mathrm{H}_{\mathfrak{m}}^{\dim_R(M)}(M) \neq 0$.
Proof:
It is well-known and not difficult to see that for every $n \in \mathbf{N}$, every ideal $I \subseteq R$ and every finitely generated R-module N the following statements are equivalent:
(i) $H^i_I(N) = 0$ for all $i \geq n$.
(ii) $H^i_I(R/\mathfrak{p}) = 0$ for all $i \geq n$ and all $\mathfrak{p} \in \mathrm{Supp}_R(N)$
This fact implies (setting $d := \dim_R(M) = \dim(R/\operatorname{ann}_R(M))$) $\mathrm{H}^d_{\mathfrak{m}}(M) \neq 0 \iff \mathrm{H}^d_{\mathfrak{m}}(R/\operatorname{ann}_R(M)) \neq 0$. Thus we may assume that $M = R$ and R is a domain. Again, we set $d := \dim(R)$ and choose a system of parameters $x_1, \ldots, x_d \in R$ for R. Theorem 3.1.3 (ii) implies $\{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^d_{\mathfrak{m}}(R)))$; in particular, $\mathrm{H}^d_{\mathfrak{m}}(R) \neq 0$.

6.1.2 Theorem

Let $(R, \mathfrak{m})$ be a noetherian local complete equicharacteristic domain, $n := \dim(R) \geq 1$ and $I \subsetneq R$ an ideal. Then

$$\mathrm{H}^n_I(R) \neq 0 \iff \sqrt{I} = \mathfrak{m}$$

holds.
Proof:
$\Longleftarrow$ follows from theorem 6.1.1. $\Longrightarrow$: By induction on n, the case $n = 1$ being trivial; we assume that $n > 1$ and that the theorem is true for smaller n. Let $\tilde{R}$ be the normalization of R. $\tilde{R}$ is a noetherian local (as R is a domain) complete equicharacteristic domain and is module-finite over R, i. e. $\dim(\tilde{R}) = \dim(R)$; we denote the maximal ideal of $\tilde{R}$ by $\mathfrak{m}_{\tilde{R}}$. One has $\mathrm{H}^n_{I\tilde{R}}(\tilde{R}) = \mathrm{H}^n_I(\tilde{R}) \neq 0$, because of $\mathrm{Supp}_R(\tilde{R}) = \mathrm{Spec}(R)$. It suffices to show $\sqrt{I\tilde{R}} = \mathfrak{m}_{\tilde{R}}$ and so we may assume that R is normal.
We choose $x_1, \ldots, x_n \in I$ such that $\sqrt{(x_1, \ldots, x_n)R} = \sqrt{I}$ and define the subring $R_0 := k[[x_1, \ldots, x_n]]$ of R, where k is any fixed coefficient field of R; by $\mathfrak{m}_0$ we denote the maximal ideal of R_0. Because of $\mathrm{H}^n_{\mathfrak{m}_0}(R) \neq 0$ we may conclude $\dim(R_0) = n$, i. e. R_0 is a formal power series ring over k in the n variables $x_1, \ldots, x_n$. We set

$$x := x_1, I_x := \{r \in R | \forall_{\varphi \in \mathrm{Hom}_{R_0}(R, R_0)} \varphi(r) \in xR_0\} \quad .$$

I_x is an ideal of R such that $R \cdot x \subseteq I_x$; furthermore, we have $I_x \subsetneq R$ because of

$$0 \neq \mathrm{Hom}_{R_0}(\mathrm{H}^n_I(R), \mathrm{E}_{R_0}(k)) \quad .$$

For every $r \in I_x$ and every $\varphi \in \mathrm{Hom}_{R_0}(R, R_0)$ we have $\mathrm{im}(r \cdot \varphi) \subseteq R_0 \cdot x$ and thus there exists $\varphi_0 \in \mathrm{Hom}_{R_0}(R, R_0)$ such that $r \cdot \varphi = x \cdot \varphi_0$. Therefore, $\frac{r}{x}$ operates in a canonical way on the finite R-module $\mathrm{Hom}_{R_0}(R, R_0)$ (note that $\mathrm{H}^n_I(R)$ is artinian as surjective image of the artinian R-module $\mathrm{H}^n_{\mathfrak{m}}(R)$; we conclude that $\frac{r}{x}$ (as an element of $Q(R)$, the quotient field of R) is integral over R; But R is normal and so we have $r \in R \cdot x$; this implies $I_x = R \cdot x$. We have

$$\mathrm{Ann}_R(\mathrm{Hom}_{R_0/xR_0}(R/I_x, R_0/xR_0)) = I_x = R \cdot x$$

and for every $\mathfrak{P} \in \mathrm{Ass}_R(\mathrm{Hom}_{R_0/xR_0}(R/I_x, R_0/xR_0))$ there exists a non-trial R_0/xR_0-linear map $R/\mathfrak{P} \to R_0/xR_0$; by an easy Matlis duality argument, we conclude $\mathrm{H}^{n-1}_I(R/\mathfrak{P}) \neq 0$ and, therefore, $\mathrm{height}(\mathfrak{P}) = 1$ and $\sqrt{I + \mathfrak{P}} = \mathfrak{m}$ (induction hypothesis). We have shown

$$\mathrm{Ass}_R(\mathrm{Hom}_{R_0/xR_0}(R/I_x, R_0/xR_0)) = \mathrm{Min}_R(R/xR)$$

and $\sqrt{I + \mathfrak{P}} = \mathfrak{m}$ for every $\mathfrak{P} \in \mathrm{Min}_R(R/xR)$. Now, because of

$$\bigcap_{\mathfrak{P} \in \mathrm{Min}_R(R/xR)} \mathfrak{P} = \sqrt{xR} \subseteq \sqrt{I}$$

it follows that $\sqrt{I} = \mathfrak{m}$.

6.1.3 Lemma
Let $(S, \mathfrak{m})$ be a noetherian local regular ring, $\mathfrak{P} \subseteq S$ a prime ideal and $f \in \mathfrak{P}$ an irreducible element. Then f operates injectively on $\widehat{S_{\mathfrak{P}}}/S$.
Proof:
We take a primary decomposition

$$f\widehat{S_{\mathfrak{P}}} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$$

of $f\widehat{S_{\mathfrak{P}}}$ such that $\mathrm{height}(\mathfrak{q}_i) = 1$ $(i = 1, \ldots, n)$. $f \in \mathfrak{P}$ implies $n \geq 1$. Clearly we have $f \in \mathfrak{q}_1 \cap S$ and $\mathfrak{q}_1 \cap S \subseteq S$ is a primary ideal of height one (the canonical map $S \to \widehat{S_{\mathfrak{P}}}$ is flat and thus going-down holds). On the other hand $fS \subseteq S$ is a prime ideal of height one. Therefore we have

$$fS \supseteq \mathfrak{q}_1 \cap S \supseteq f\widehat{S_{\mathfrak{P}}} \cap S$$

and hence $fS = f\widehat{S_{\mathfrak{P}}} \cap S$. The statement follows.

6.1.4 Theorem
Let $(R, \mathfrak{m})$ be a noetherian local complete domain and $I \subseteq R$ an ideal such that $\sqrt{I} \subsetneq \mathfrak{m}$. Then

$$\mathrm{H}^{\dim(R)}_I(R) = 0$$

holds.
Proof:

Set $n := \dim(R)$. For proper ideals $I_1 \subseteq I_2$ of R the canonical map $\mathrm{H}^n_{I_2}(R) \to \mathrm{H}^n_{I_1}(R)$ is surjective and so we may assume I is a prime ideal of R of height $n-1$. Now we choose $S, \mathfrak{Q}$ and ρ as in lemma 3.2.2. Then we have

$$\mathrm{H}^{\dim(R)}_I(R) = \mathrm{H}^{\dim(R)}_{\mathfrak{Q}}(S/fS) \otimes_{S/fS} R$$

(f is a generator of the height one prime ideal $\ker(\rho)$. Lemmas 3.2.1 and 6.3.1 imply that f operates injectively on $D_S(\mathrm{H}^{\dim(R)}_{\mathfrak{Q}}(S))$ and thus the statement follows.

6.2 Generalization of an example of Hartshorne

The idea of this subsection is that, by Theorem 4.3.4, the Matlis duals of certain local cohomology modules have infinitely many associated prime ideals; but then this local cohomology module can not be artinian. It turns out that this leads to a generalization of an example of Hartshorne ([Ha1, section 3]); more details on this generalization can be found in [HS2, section 1]. The author thanks Gennady Lyubeznik for drawing his attention to this example.

6.2.1 Example

Let k be a field, $R = k[[X_1, X_2, X_3, X_4]]$ a power series algebra over k in four variables, $I = (X_1, X_2)R$ and, for every $\lambda \in k$, define

$$\mathfrak{p}_\lambda := (X_3 + \lambda X_1, X_4 + \lambda X_2)R \quad .$$

Clearly, every $\mathfrak{p}_\lambda$ is a height two prime ideal of R and, by theorem 3.1.3 (v), is associated to $D = D(\mathrm{H}^2_I(R))$. On the other hand, for every $\lambda \in k$, one has

$$p := X_1X_4 + X_2X_3 \in \mathfrak{p}_\lambda$$

(because of $p = X_1(X_4 - \lambda X_2) + X_2(X_3 + \lambda X_1)$). Therefore, at least if k is infinite, D has infinitely many associated primes containing p. This implies that

$$\mathrm{Hom}_R(R/pR, D)$$

cannot be finitely generated. But $\mathrm{Hom}_R(R/pR, D)$ is the Matlis dual of

$$\mathrm{H}^2_I(R/pR)$$

and so $\mathrm{H}^2_I(R/pR)$ cannot be artinian.

6.2.2 Remark

This is essentially Hartshorne's example ([Ha1, section 3]), the main difference is that Hartshorne works over a ring of the form $k[X_3, X_4][[X_1, X_2]]$, while we work over a ring of the form $k[[X_1, X_2, X_3, X_4]]$; but the two versions are essentially the same, because the module

$$\mathrm{H}^2_{(X_1,X_2)}(k[X_3, X_4][[X_1, X_2]]/(X_1X_4 + X_2X_3))$$

is naturally a module over $k[[X_1, X_2, X_3, X_4]]$, because its support is $\{(X_1, X_2, X_3, X_4)\}$. This is true, because for every prime ideal $\mathfrak{p} \neq (X_1, X_2, X_3, X_4)$ of $k[X_3, X_4][[X_1, X_2]]$ containing $X_1X_4 + X_2X_3$ the ring $(k[X_3, X_4][[X_1, X_2]]/(X_1X_4+X_2X_3))_{\mathfrak{p}}$ is regular, and so Hartshorne-Lichtenbaum vanishing shows that $\mathrm{H}^2_{(X_1,X_2)}(k[X_3, X_4][[X_1, X_2]]/(X_1X_4 + X_2X_3))_{\mathfrak{p}} = 0$.

A similar technique like in the example above works to show that $\mathrm{H}^{n-2}_I(R/pR)$ is not artinian in the general situation $R = k[[X_1, \ldots, X_n]]$, $n \geq 4$, $I = (X_1, \ldots, X_{n-2})R$ and $p \in R$ a prime element such that $p \in (X_{n-1}, X_n)R$, even if the field k is finite:

6.2.3 Theorem
Let k be a field, $n \geq 4$, $R = k[[X_1, \ldots, X_n]]$, $I = (X_1, \ldots, X_{n-2})R$ and $p \in R$ a prime element such that $p \in (X_{n-1}, X_n)R$. Then $\mathrm{H}^{n-2}_I(R/pR)$ is not artinian.
Proof:
Set $D := D(\mathrm{H}^{n-2}_I(R))$. If $p \notin I$, it is easy to see that

$$\mathrm{Supp}_R(\mathrm{H}^{n-2}_I(R/pR)) = \mathcal{V}(I + pR) \quad ,$$

the set of prime ideals of R containing $I+pR$, and so $\mathrm{H}^{n-2}_I(R/pR)$ is not artinian (it is not zero-dimensional). We assume $p \in I$: If $\mathrm{H}^{n-2}_I(R/pR)$ was artinian, $D(\mathrm{H}^{n-2}_I(R/pR))$ would be finitely generated; but we have seen before that, because of the exactness of D and the right-exactness of H^{n-2}_I,

$$D(\mathrm{H}^{n-2}_I(R/pR)) = \mathrm{Hom}_R(R/pR, D) \quad ,$$

and from Theorem 4.3.4 we know that the latter module is not finitely generated (it has infinitely many associated prime ideals).

6.2.4 Remark Marley and Vassilev have shown

Theorem ([MV, theorem 2.3])
Let $(T, \mathfrak{m})$ be a noetherian local ring of dimension at least two. Let $R = T[x_1, \ldots, x_n]$ be a polynomial ring in n variables over T, $I = (x_1, \ldots, x_n)$, and $f \in R$ a homogenous polynomial whose coefficients form a system of parameters for T. Then the *socle of $\mathrm{H}^n_I(R/fR)$ is infinite dimensional.

In their paper [MV], Marley and Vassilev say (in section 1) that Hartshorne's example is obtained by letting $T = k[[u, v]], n = 2$ and $f = ux + vy$; there is a slight difference between the two situations that comes from the fact that Hartshorne works over a ring of the form $k[x, y][[u, v]]$ while Marley and Vassilev work over a ring of the form $k[[u, v]][x, y]$. The two rings are not the same. But, as

$$\mathrm{Supp}_R(\mathrm{H}^2_{(u,v)}(R/(uy + vx))) = \{(x, y, u, v)\}$$

(both for $R = k[x, y][[u, v]]$ and for $R = k[[u, v]][x, y]$), the local cohomology module in question is (in both cases) naturally a module over $k[[x, y, u, v]]$ and, therefore, both versions are equivalent, i. e. the result of Marley and Vassilev is a generalization of Hartshorne's example.

6.2.5 Remark
[MV, theorem 2.3] and theorem 6.2.3 are both generalizations of Hartshorne's example, but, due to different hypotheses, they can only be compared in the following special case: k a field, $n \geq 4$,

$$R_0 = k[[X_{n-1}, X_n]][X_1, \ldots, X_{n-2}] \quad ,$$

$$R = k[[X_1, \dots, X_n]] \quad ,$$

$I = (X_1, \dots, X_{n-2})R$, $p \in R_0$ a homogenous element such that p is prime as an element of R. Then [MV, theorem 2.3] says (implicitly) that

$$\mathrm{H}_I^{n-2}(R/pR)$$

is not artinian, if the coefficients of $p \in R_0$ in $k[[X_{n-1}, X_n]]$ form a system of parameters in $k[[X_{n-1}, X_n]]$, while theorem 6.2.3 says that the same module is not artinian if none of these coefficients of p is a unit in $k[[X_{n-1}, X_n]]$.

6.3 A necessary condition for set-theoretic complete intersections

Let $(R, \mathfrak{m})$ be a noetherian local ring and $I = (x_1, \dots, x_i)R = I \subseteq R$ a set-theoretic complete intersection ideal (in the sense that its height is i). Then $\mathrm{H}_I^i(R) \neq 0$ (this can be seen by localizing at a height-i prime ideal of R containing I). On the other hand, statement (ii) from theorem 6.3.1 below presents a necessary condition for $\mathrm{H}_I^i(R) \neq 0$.

6.3.1 Theorem
Let $(R, \mathfrak{m})$ be a noetherian local complete domain containing a field k and $x_1, \dots, x_i$ a sequence in R ($i \geq 1$). Define $R_0 := k[[x_1, \dots, x_i]]$ as a subring of R.
(i) The following two statements are equivalent:
(α) $\mathrm{H}^i_{(x_1,\dots,x_i)R}(R) \neq 0$.
(β) $\mathrm{Hom}_{R_0}(R, R_0) \neq 0$ and $\dim(R_0) = i$.
(ii) If the equivalent statements of (i) hold, one has

$$R \cap Q(R_0) = R_0 \quad ,$$

where $Q(R_0)$ denotes the quotient field of R_0 and where the intersection is taken in the quotient field $Q(R)$ of R.
Proof:
We denote the maximal ideal of R_0 by $\mathfrak{m}_0$.
(i) We have

$$\mathrm{H}^i_{(x_1,\dots,x_i)R}(R) = R \otimes_{R_0} \mathrm{H}^i_{(x_1,\dots,x_i)R_0}(R_0)$$

and, thus, $\mathrm{H}^i_{(x_1,\dots,x_i)R_0}(R_0) \neq 0$ implies $\dim(R_0) = i$; therefore we may assume that R_0 is a formal power series ring in $x_1, \dots, x_i$. Therefore, we may assume that R_0 is i-dimensional. Let $\mathrm{E}_{R_0}(k)$ be a fixed R_0-injective hull of R_0. We have

$$\begin{aligned}\mathrm{Hom}_{R_0}(\mathrm{H}^i_{(x_1,\dots,x_i)R}(R), \mathrm{E}_{R_0}(k)) &= \mathrm{Hom}_{R_0}(R \otimes_{R_0} \mathrm{H}^i_{(x_1,\dots,x_i)R_0}(R_0), \mathrm{E}_{R_0}(k)) \\ &= \mathrm{Hom}_{R_0}(R, \mathrm{Hom}_{R_0}(\mathrm{H}^i_{(x_1,\dots,x_i)R_0}(R_0), \mathrm{E}_{R_0}(k))) \\ &= \mathrm{Hom}_{R_0}(R, R_0) \quad ,\end{aligned}$$

where we have used the fact that R_0 is a formal power series ring in $x_1, \dots, x_i$ over k. This identity shows the stated equivalence.
(ii) Under the given assumptions, we have $\mathrm{Hom}_{R_0}(R, R_0) \neq 0$. Let

$$\varphi \in \mathrm{Hom}_{R_0}(R, R_0)$$

be any non-zero element and let $r_0 \in R_0 \setminus \{0\}, r \in R$ such that $r_0 \cdot r \in R_0$. We have to show $r \in R_0$: We set

$$r_0' := r_0 r$$

and conclude

$$r_0 \varphi(r) = \varphi(r_0') = r_0' \varphi(1) \quad .$$

This shows

$$\varphi(1) r = \varphi(1) \frac{r_0'}{r_0} = \varphi(r) \in R_0 \quad .$$

On the other hand, we have

$$r_0'^2 = r_0^2 r^2$$

and thus

$$r_0^2 \varphi(r^2) = r_0'^2 \varphi(1)$$

and

$$\varphi(1) r^2 = \varphi(1) \frac{r_0'^2}{r_0^2} = \varphi(r^2) \in R_0 \quad .$$

Continuing in the same way, one sees that, for every $l \geq 1$, one has

$$\varphi(1) r^l \in R_0 \quad .$$

But this implies that the R_0-module

$$\varphi(1) \cdot < 1, r, r^2, \ldots >_{R_0}$$

is finitely generated ($< 1, r, r^2, \ldots >_{R_0}$ stands for the R_0-submodule of R generated by $1, r, r^2, \ldots$). But, as R is a domain,

$$< 1, r, r^2, \ldots > R_0$$

is then finitely generated, too, i. e. r is necessarily contained in R_0.

6.4 A generalization of local duality

Over some rings (e. g. over complete Cohen-Macaulay rings), there is a correspondence between certain Ext-modules on the one hand and certain local cohomology modules on the other hand; this correspondence is given (in both directions) by taking the Matlis dual and is called local duality. This result can e. g. be found in [BS, section 11]. In the form in which it is usually presented, local duality works only if the support ideal is $\mathfrak{m}$, i. e. if one takes local cohomology with support in $\mathfrak{m}$. But, below we generalize this result to a large class of support ideals I.

6.4.1 Theorem

Let $(R, \mathfrak{m})$ be a noetherian local ring, $I \subseteq R$ an ideal, $h \in \mathbb{N}$ such that

$$\mathrm{H}_I^l(R) \neq 0 \iff l = h$$

holds and M an R-module. Then, for every $i \in \{0, \ldots, h\}$, one has

$$\mathrm{Ext}_R^i(M, D(\mathrm{H}_I^h(R))) = D(\mathrm{H}_I^{h-i}(M)) \quad .$$

Proof:
We take the sequence of functors $(D \circ \mathrm{H}_I^{h-i})_{i \in \mathrm{N}}$ from the category of R-modules to itself; of course, $\mathrm{H}_I^M = 0$ for every $M < 0$. Given a short exact sequence of R-modules

$$0 \to M' \to M \to M'' \to 0 \quad ,$$

we clearly get an exact sequence of the form

$$\begin{aligned} 0 \to D(\mathrm{H}_I^h(M'')) \to D(\mathrm{H}_I^h(M)) \to D(\mathrm{H}_I^h(M')) \to \\ \to D(\mathrm{H}_I^{h-1}(M'')) \to \dots \end{aligned}$$

(note that, by our hypothesis, $\mathrm{H}_I^{h+1}(M') = M' \otimes_R \mathrm{H}_I^{h+1}(R) = 0$). In the case $i = 0$ we get, for any R-module M,

$$\begin{aligned} D(\mathrm{H}_I^h(M)) &= \mathrm{Hom}_R(\mathrm{H}_I^h(M), \mathrm{E}_R(R/\mathfrak{m})) \\ &= \mathrm{Hom}_R(M \otimes_R \mathrm{H}_I^h(R), \mathrm{E}_R(R/\mathfrak{m})) \\ &= \mathrm{Hom}_R(M, \mathrm{Hom}_R(\mathrm{H}_I^h(R), \mathrm{E}_R(R/\mathfrak{m}))) \\ &= \mathrm{Hom}_R(M, D(\mathrm{H}_I^h(R))) \quad . \end{aligned}$$

Finally, for every $i > 0$ and every $m \in \mathrm{N}$, we have $\mathrm{H}_I^{h-i}(R^m) = 0$ and hence $\mathrm{H}_I^{h-i}(F) = 0$ for every free R-module F; we get

$$D(\mathrm{H}_I^{h-i}(F)) = 0$$

for every $i > 0$ and every free R-module F. By some well-known homology theory, the last three properties imply our statement.

6.4.2 Remark

If, in the situation of theorem 6.4.1, R is complete and Cohen-Macaulay and $I = \mathfrak{m}$ (then $h = \dim(R)$ necessarily), the statement takes the form

$$\mathrm{Ext}_R^i(M, D(\mathrm{H}_\mathfrak{m}^{\dim(R)}(R))) = D(\mathrm{H}_\mathfrak{m}^{\dim(R)-i}(M))$$

for every R-module M and every $i \in \{0, \dots, \dim(R)\}$. If we assume furthermore that M is finitely generated, then $\mathrm{H}_\mathfrak{m}^{\dim(R)-i}(M)$ is artinian and the above statement implies (in fact, is equivalent to)

$$D(\mathrm{Ext}_R^i(M, D(\mathrm{H}_\mathfrak{m}^{\dim(R)}(R)))) = \mathrm{H}_\mathfrak{m}^{\dim(R)-i}(M) \quad .$$

We study the R-module $D(\mathrm{H}_\mathfrak{m}^{\dim(R)}(R))$: By Matlis duality, it is finitely generated. We calculate its type, which is defined as the following $R/\mathfrak{m}$-vector space dimension:

$$\dim_{R/\mathfrak{m}}(\mathrm{Ext}_R^{\dim(R)}(R/\mathfrak{m}, D(_\mathfrak{m}^{\dim(R)}(R)))) = \dim_{R/\mathfrak{m}}(D(\mathrm{H}_\mathfrak{m}^0(R/\mathfrak{m}))) = 1$$

(note that, for the first equality, we use theorem 6.4.1 again). The $\mathfrak{m}$-depth of $D(\mathrm{H}_\mathfrak{m}^{\dim(R)}(R))$ is $\dim(R)$ (this follows from theorem 1.1.2, take any parameter sequence $\underline{x}$ of R, it will be a regular sequence on $D(\mathrm{H}_\mathfrak{m}^{\dim(R)}(R))$), i. e. $D(\mathrm{H}_\mathfrak{m}^{\dim(R)}(R))$ is a maximal Cohen-Macaulay module (over R). By definition, these properties show that

$$D(\mathrm{H}_\mathfrak{m}^{\dim(R)}(R)) =: \omega_R$$

is a canonical module for R, and the statement of theorem 6.4.1 becomes

$$D(\mathrm{Ext}_R^i(M, \omega_R)) = \mathrm{H}_\mathfrak{m}^{\dim(R)-i}(M) \quad .$$

Clearly, this is a version of the local duality theorem (see, e. g., [BS, section 11] for more details on local duality).

7 Further Topics

7.1 Local Cohomology of formal schemes

In some cases we can consider the Matlis duals of local cohomology modules as certain local cohomology modules of the structure sheaf of some formal scheme (see [Og, in particular section 2]), here are the details:

7.1.1 Remark

Let $(R, \mathfrak{m})$ be a noetherian local Gorenstein ring and I an ideal of R. We define

$$X := \operatorname{Spec}(R), Y := \mathcal{V}(I) \quad ,$$

i. e. Y is the closed subscheme of X defined by I. We denote by $\mathcal{X}$ the formal completion of X along Y and by p the closed point of the topological space underlying $\mathcal{X}$ (note that as topological spaces $\mathcal{X}$ and X are the same). Then, for every $i \in \mathbb{N}$, there is a canonical isomorphism

$$\mathrm{H}^i_p(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = D(\mathrm{H}_I^{\dim(R)-i}(R))$$

(see [Og, 2.2.3]). This follows essentially from local duality, the fact that $\mathrm{H}^i_{\mathfrak{m}}(R/I^v)$ is artinian for every $v \geq 1$ and the existence of a short exact sequence

$$0 \to R^1 \operatorname{invlim}_v \mathrm{H}^{i-1}_{\mathfrak{m}}(R/I^v) \to \mathrm{H}^i_p(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \operatorname{invlim}_v \mathrm{H}^i_{\mathfrak{m}}(R/I^v) \to 0 \quad .$$

Note that local cohomology of a (formal) coherent sheaf on a formal scheme is defined in the sense of Grothendieck, i. e. as (local) cohomology of an abelian sheaf on a topological space.

7.2 $D(\mathrm{H}^i_I(R))$ has a natural D-module structure

Let k be a field and $R = k[[X_1, \dots X_n]]$ a power series ring over k in n variables. Let

$$D(R, k) \subseteq \operatorname{End}_k(R)$$

be the (non-commutative) subring defined by the multiplication maps by $r \in R$ (for all $r \in R$) and by all k-linear derivation maps from R to R. $D := D(R, k)$ is the so-called ring of k-linear differential operators on R. [Bj] contains material on the ring $D(R, k)$ and on similar rings; D-modules in relation with local cohomology modules have been studied in [Ly1]. For $i = 1, \dots, n$ let ∂_i denote the partial derivation map from R to R with respect to X_i. Then, as an R-module, one has

$$D(R, k) = \oplus_{i_1, \dots, i_n \in \mathbb{N}} R \cdot \partial_1^{i_1} \dots \partial_n^{i_n} \quad . \tag{1}$$

Now, let $I \subseteq R$ be an ideal and $i \in \mathbb{N}$. We will demonstrate that there is a canonical left-D-module structure on $D(\mathrm{H}^i_I(R))$ (the following idea was inspired by Gennady Lyubeznik). To do so, by identity (1), it is sufficient to determine the action of an arbitrary k-linear derivation $\delta : R \to R$ on $D(\mathrm{H}^i_I(R))$, to extend it to an action of $D(R, k)$ on $D(\mathrm{H}^i_I(R))$ and to show that this action is well-defined and satisfies all axioms of a left-D-module. The derivation δ induces a k-linear map

$$R/I^v \to R/I^{v-1} \quad (v \geq 1)$$

and, in a canonical way, a map of complexes from the Čech complex of R/I^v with respect to $X_1, \ldots, X_n$ to the Čech complex of R/I^{v-1} with respect to $X_1, \ldots, X_n$ $(v \geq 1)$. By taking cohomology, we get a map

$$\mathrm{H}^{n-i}_{\mathfrak{m}}(R/I^v) \to \mathrm{H}^{n-i}_{\mathfrak{m}}(R/I^{v-1}) \ \ (v \geq 1) \ ,$$

where $\mathfrak{m}$ stands for the maximal ideal of R. These maps induce a map

$$\mathrm{invlim}_{v\in\mathrm{N}}(\mathrm{H}^{n-i}_{\mathfrak{m}}(R/I^v)) \to \mathrm{invlim}_{v\in\mathrm{N}}(\mathrm{H}^{n-i}_{\mathfrak{m}}(R/I^v))$$

(note that the maps of the above inverse system are induced by the canonical epimorphisms $R/I^v \to R/I^{v-1}$). But, by local duality and $\mathrm{H}^i_I(R) = \mathrm{dirlim}_{v\in\mathrm{N}}(Ext^i_R(R/I^v, R))$, one has

$$\mathrm{invlim}_{v\in\mathrm{N}}(\mathrm{H}^{n-i}_{\mathfrak{m}}(R/I^v)) = D(\mathrm{dirlim}_{v\in\mathrm{N}}(Ext^i_R(R/I^v, R))) = D(\mathrm{H}^i_I(R)) \ \ .$$

Now, having determined the action of the element δ on $D(\mathrm{H}^i_I(R))$, by (1) it is clear how to extend this to an action of $D(R,k)$ on $D(\mathrm{H}^i_I(R))$ such that $D(\mathrm{H}^i_I(R))$ becomes a left-D-module (note that, for every k-linear derivation $\delta : R \to R$ and every $r \in R$, we have $\delta(r \cdot d) = \delta(r) \cdot d + r \cdot \delta(d)$, i. e. the action of D on $D(\mathrm{H}^i_I(R)$ makes it a left-D-module). We have seen (in various situations) in sections 2, 3 and 4, that, in general,

$$D(\mathrm{H}^i_I(R))$$

has infinitely many associated primes. On the other hand, one knows from [Ly1, Theorem 2.4 (c)] (at least if $\mathrm{char}(k) = 0$), that every finitely generated left-D-module has only finitely many associated prime ideals (as R-module, of course). This shows that, in general, $D(\mathrm{H}^i_I(R))$ is an example of a non-finitely generated left-D-module. In particular, $D(\mathrm{H}^i_I(R))$ is not holonomic in general (see [Bj] for the notion of holonomic modules).

7.3 The zeroth Bass number of $D(\mathrm{H}^i_I(R))$ (w. r. t. the zero ideal) is not finite in general

Let $(R, \mathfrak{m})$ be a noetherian local domain, $i \geq 1$ and $x_1, \ldots, x_i \in R$. Then, as we have seen in theorem 3.1.3 (ii), one has

$$\{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)))$$

in some situations; actually, if conjecture (*) holds, this is true provided $\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R) \neq 0$ holds. It is natural to ask for the associated Bass number of $D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))$, i. e. the $Q(R)$-vector space dimension of

$$D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)) \otimes_R Q(R) \ \ ,$$

where $Q(R)$ stands for the quotient field of R. As we will see below, this number is not finite in general; more precisely, we consider the following case: Let k be a field, $R = k[[X_1, \ldots, X_n]]$ a power series algebra over k in $n \geq 2$ variables, $1 \leq i < n$ and I the ideal $(X_1, \ldots, X_i)R$ of R; in this situation

$$\dim_{Q(R)}(D(\mathrm{H}^i_I(R)) \otimes_R Q(R)) = \infty$$

holds, see theorem 7.3.2 below for a proof.

7.3.1 Remark

Note that in section 4.3 (see, in particular, the proof of remark 4.3.6) we introduced some notation on polynomials in "inverse variables" and we explained and proved (note that the situation here is more general then in remark 4.3.6, where i was $n-2$, but the proof of 4.3.6 works in this more general situation too) the following formulas:

$$\mathrm{H}^i_I(R) = k[[X_{i+1},\dots,X_n]][X_1^{-1},\dots,X_i^{-1}] \quad ,$$

$$\mathrm{E}_R(k) = k[X_1^{-1},\dots,X_n^{-1}]$$

and

$$D(\mathrm{H}^i_I(R)) = k[X_{i+1}^{-1},\dots,X_n^{-1}][[X_1,\dots,X_i]] \quad .$$

Also note that the latter module is different from and larger than the module

$$k[[X_1,\dots,X_i]][X_{i+1}^{-1},\dots,X_n^{-1}] \quad .$$

The following proof is technical; its basic idea is the following one: Let k be a field, $R = k[[X,Y]]$ a power series algebra over k in two variables; then we have

$$\mathrm{H}^1_{XR}(R) = k[[Y]][X^{-1}]$$

and

$$D := D(\mathrm{H}^1_{XR}(R)) = k[Y^{-1}][[X]] \quad .$$

Set

$$\begin{aligned} d_2 &:= \sum_{l\in\mathrm{N}} Y^{-l^2}X^l \\ &= 1 + Y^{-1}X + Y^{-4}X^2 + Y^{-9}X^3 + \ldots \in D \end{aligned}$$

and let $r \in R\setminus\{0\}$ be arbitrary. Because of $r \neq 0$ we can write

$$r = X^{a+1}\cdot h + X^a \cdot g$$

with some $h \in R, g \in k[[Y]]\setminus\{0\}$. Then, at least for $l >> 0$, the coefficient of $r\cdot d_2$ in front of X^l is

$$h^*\cdot Y^{-(l-a-1)^2} + g\cdot Y^{-(l-a)^2}$$

for some $h^* \in k[[Y]]$. Now, if we write

$$g = c_bY^b + c_{b+1}Y^{b+1} + \ldots$$

for some $b \in \mathrm{N}, c_b \neq 0$ and observe the fact

$$-(l-a)^2 + b < -(l-a-1)^2 \quad (l >> 0) \quad ,$$

it follows that the term

$$c_b \cdot Y^{-(l-a)^2+b}$$

(coming from $h^*\cdot Y^{-(l-a-1)^2} + g\cdot Y^{-(l-a)^2}$) cannot be canceled out by any other term. In fact, for $l >> 0$, the lowest non-vanishing Y-exponent of the coefficient in front of X^l, is $-(l-a)^2 + b$. The crucial point is that the sequences $-(l-a)^2+b$ and $-l^2$ agree up to the two shifts given by a and b. This means that some information about d_2 is stored in rd_2.

7.3.2 Theorem

Let k be a field, $R = k[[X_1, \ldots, X_n]]$ a power series algebra over k in $n \geq 2$ variables, $1 \leq i < n$ and I the ideal $(X_1, \ldots, X_i)R$ of R. Then

$$\dim_{Q(R)}(D(\mathrm{H}^i_I(R)) \otimes_R Q(R)) = \infty$$

holds.

Proof:

As the proof is technical we will first show the case $n = 2, i = 1$; in the remark after this proof we will explain how one can reduce the general to this special case. Set $X = X_1, Y = X_2$ and

$$D := D(\mathrm{H}^2_I(R)) = k[Y^{-1}][[X]] \quad .$$

For every $n \in \mathrm{N} \setminus \{0\}$, set

$$d_n := \sum_{l \in \mathrm{N}} Y^{-l^n} \cdot X^l \in D \quad .$$

It is sufficient to show the following statement: The elements $(d_n \otimes 1)_{n \in \mathrm{N} \setminus \{0\}}$ in $D \otimes_R Q(R)$ are $Q(R)$-linear independent:

We define an equivalence relation on $\mathbf{Z}^{\mathrm{N}}$ (the set of all maps from N to $\mathbf{Z}$, i. e. infinite sequences of integers) by saying that $(a_n), (b_n) \in \mathbf{Z}^{\mathrm{N}}$ are equivalent (short form: $(a_n) \sim (b_n)$) iff there exist $N, M \in \mathrm{N}$ and $p \in \mathbf{Z}$ such that

$$a_{N+1} = b_{M+1} + p, a_{N+2} = b_{M+2} + p, \ldots$$

hold. It is easy to see that $\sim$ is an equivalence relation on $\mathbf{Z}^{\mathrm{N}}$. For every $d \in D$, we define $\delta(d) \in \mathbf{Z}^{\mathrm{N}}$ in the following way: Let $f_l \in k[Y^{-1}]$ be the coefficient of d in front of X^l; we set

$$(\delta(d))(l) := 0$$

if $f_l = 0$ and

$$(\delta(d))(l) := s$$

if s is the smallest Y-exponent of f_l, i. e.

$$f_l = c_s Y^s + c_{s+1} Y^{s+1} + \ldots + c_0 \cdot 1$$

for some $c_s \neq 0$.

Now suppose that $r_1, \ldots, r_{n_0} \in R$ are given such that $r_{n_0} \neq 0$. We claim that

$$\delta(r_1 d_1 + \cdot + r_{n_0} d_{n_0}) \sim \delta(d_{n_0})$$

holds. Note that if we prove this statement we are done, essentially because then $r_1 d_1 + \ldots + r_{n_0} d_{n_0}$ can not be zero.

It is obvious that one has $\delta(d+d') \sim \delta(d_{N_2})$ for given $d, d' \in D$ such that

$$\delta(d) \sim \delta(d_{N_1}), \delta(d') \sim \delta(d_{N_2}), N_2 > N_1 \quad .$$

For this reason it is even sufficient to prove the following statement: For a fixed $n \in \mathbf{N} \setminus \{0\}$ and for any $r \in R \setminus \{0\}$ one has

$$\delta(rd_n) \sim \delta(d_n) \quad .$$

We can write

$$r = X^{a+1} \cdot h + X^a \cdot g$$

with $a \in \mathbf{N}, h \in k[[X,Y]]$ and $g \in k[[Y]] \setminus \{0\}$. We get

$$\delta(r \cdot d_n) \sim \delta(\sum_{l \geq a+1} (hY^{-(l-a-1)^n} + gY^{-(l-a)^n})X^l)$$

and we write

$$g = c_b Y^b + c_{b+1} Y^{b+1} + \dots$$

with $c_b \in k^*$. Now, because of

$$-(l-a)^n + b < -(l-a-1)^n \quad (l >> 0)$$

it is clear that, for $l >> 0$, the smallest Y-exponent in front of X^l (of the power series $r \cdot d_n$) is $-(l-a)^n + b$. Therefore, one has

$$\delta(r \cdot d_n) \sim (-l^n) \sim \delta(d_n)$$

and we are done.

7.3.3 Remark

A proof of the general case of theorem can be obtained e. g. in the following way: First, we use theorem 3.1.2 repeatedly to get a surjection

$$\mathrm{H}^i_{(X_1,\dots,X_i)R}(R) \to \mathrm{H}^{n-1}_{(X_1,\dots,X_{n-1})R}(R)$$

and hence an injection

$$D(\mathrm{H}^{n-1}_{(X_1,\dots,X_{n-1})R}(R)) \to D(\mathrm{H}^i_{(X_1,\dots,X_i)R}(R)) \quad ,$$

which allows us to reduce to the case $i = n-1$; then it is possible to adapt our proof of theorem 7.3.2 with some minor changes: Instead of working with maps $\mathbf{N} \to \mathbf{Z}$, one works with maps

$$\mathbf{N}^{n-1} \to \mathbf{Z}$$

and also with multi-indices instead of indices.

7.4 On the module $\mathrm{H}^h_I(D(\mathrm{H}^h_I(R)))$

In the previous section we were interested in modules of the form

$$D := D(\mathrm{H}^i_I(R)) \quad ,$$

where I is an ideal in a local ring R. In this section we compute the local cohomology module

$$\mathrm{H}^i_I(D) \quad .$$

Our results say (essentially) that this module is $\mathrm{E}_R(R/\mathfrak{m})$ if I is a set-theoretic complete intersection and it is either $\mathrm{E}_R(R/\mathfrak{m})$ or zero in general (see theorems 7.4.1 and 7.4.2 for precise formulations and proofs).

7.4.1 Theorem

Let $(R,\mathfrak{m})$ be a noetherian local complete Cohen-Macaulay ring with coefficient field k and $x_1,\ldots,x_i \in R$ $(i \geq 1)$ a regular sequence in R. Set $I := (x_1,\ldots,x_i)R)$ (I is a set-theoretic complete intersection ideal of R). Then one has

$$\mathrm{H}^i_I(D(\mathrm{H}^i_I(R))) = \mathrm{E}_R(k) \quad .$$

Proof:

First we show a special case: Assume that $R = k[[X_1,\ldots,X_n]]$ is a formal power series algebra over k in n variables and $x_1 = X_1,\ldots,x_i = X_i$. Then, as we have seen in the proof of remark 4.3.6 (note that the situation here is more general then in remark 4.3.6, where i was $n-2$, but the proof of 4.3.6 works in this more general situation too), we have

$$\mathrm{H}^i_I(R) = k[[X_{i+1},\ldots,X_n]][X_1^{-1},\ldots,X_i^{-1}]$$

and

$$D(\mathrm{H}^i_I(R)) = k[X_{i+1}^{-1},\ldots,X_n^{-1}][[X_1,\ldots,X_i]]$$

(again, see section 4.3, in particular the proof of remark 4.3.6 for the notation). As the functor H^i_I is right-exact, we have

$$\begin{aligned}\mathrm{H}^i_I(D(\mathrm{H}^i_I(R))) &= \mathrm{H}^i_I(R) \otimes_R D(\mathrm{H}^i_I(R)) \\ &= k[[X_{i+1},\ldots,X_n]][X_1^{-1},\ldots,X_i^{-1}] \otimes_R k[X_{i+1}^{-1},\ldots X_n^{-1}][[X_1,\ldots,X_i]] \\ &\overset{(*)}{=} k[X_1^{-1},\ldots,X_n^{-1}] \\ &= \mathrm{E}_R(k)\end{aligned}$$

Proof of equality (*): The map

$$k[X_{i+1},\ldots,X_n][X_1^{-1},\ldots,X_i^{-1}] \otimes k[X_{i+1}^{-1},\ldots X_n^{-1}][X_1,\ldots,X_i] \to k[X_1^{-1},\ldots,X_n^{-1}]$$

$$X_{i+1}^{r_{i+1}} \cdot \ldots \cdot X_n^{r_n} \cdot X_1^{-s_1} \cdot \ldots X_i^{-s_i} \otimes X_{i+1}^{-t_{i+1}} \cdot \ldots \cdot X_n^{-t_n} \cdot X_1^{u_1} \cdot \ldots \cdot X_i^{u_i} \mapsto$$

$$\mapsto X_{i+1}^{r_{i+1}-t_{i+1}} \cdot X_n^{r_n-t_n} \cdot X_1^{u_1-s_1} \cdot \ldots \cdot X_i^{u_i-s_i} \text{ if } r_{i+1}-t_{i+1},\ldots,r_n-t_n,u_1-s_1,\ldots,u_i-s_i \leq 0$$

and to zero otherwise, induces an R-linear map

(1) $$k[[X_{i+1},\ldots,X_n]][X_1^{-1},\ldots,X_i^{-1}] \otimes_R k[X_{i+1}^{-1},\ldots X_n^{-1}][[X_1,\ldots,X_i]] \to k[X_1^{-1},\ldots,X_n^{-1}] \quad ,$$

which is surjective and maps the k-vector space generating system

$$\{X_1^{-s_1} \cdot \ldots \cdot X_i^{-s_i} \otimes X_{i+1}^{-t_{i+1}} \cdot \ldots \cdot X_n^{-t_n} | s_1,\ldots,s_i,t_{i+1},\ldots,t_n \geq 0\}$$

of the vector space on the left side of (1) to the k-basis

$$\{X_1^{-s_1} \cdot \ldots \cdot X_i^{-s_i} \cdot X_{i+1}^{-t_{i+1}} \cdot \ldots \cdot X_n^{-t_n} | s_1, \ldots, s_i, t_{i+1}, \ldots, t_n \geq 0\}$$

of the vector space on the right side of (1), and therefore provides us with the desired isomorphism in our special case.
We come to the general case: Choose $x_{i+1}, \ldots, x_n \in R$ such that $\sqrt{(x_1, \ldots, x_n)R} = \mathfrak{m}$ ($x_1, \ldots x_n$ is a s. o. p. of R). Define

$$R_0 := k[[x_1, \ldots, x_n]] \subseteq R$$

R_0 is regular of dimension n and R is a finite-rank free R_0-module. Define $I_0 := (x_1, \ldots, x_i)R_0$. We have

$$\mathrm{H}_I^i(D(\mathrm{H}_I^i(R))) = \mathrm{H}_I^i(R) \otimes_R D(\mathrm{H}_I^i(R))$$

and

$$\mathrm{H}_I^i(R) = \mathrm{H}_{I_0}^i(R_0) \otimes_{R_0} R$$

and

$$\begin{aligned} D(\mathrm{H}_I^i(R))) &= \mathrm{Hom}_R(\mathrm{H}_{I_0}^i(R_0) \otimes_{R_0} R, \mathrm{E}_R(k)) \\ &= \mathrm{Hom}_{R_0}(\mathrm{H}_{I_0}^i(R_0), \mathrm{E}_R(k)) \\ &\overset{(2)}{=} \mathrm{Hom}_{R_0}(\mathrm{H}_{I_0}^i(R_0), \mathrm{Hom}_{R_0}(R, \mathrm{E}_{R_0}(k))) \\ &= \mathrm{Hom}_{R_0}(R, D_{R_0}(\mathrm{H}_{I_0}^i(R_0))) \end{aligned}$$

For (2) we use the fact

$$\mathrm{E}_R(k) = \mathrm{Hom}_{R_0}(R, \mathrm{E}_{R_0}(k))$$

We get

$$\begin{aligned} \mathrm{H}_I^i(D(\mathrm{H}_I^i(R))) &= \mathrm{H}_{I_0}^i(R_0) \otimes_{R_0} \mathrm{Hom}_{R_0}(R, D_{R_0}(\mathrm{H}_{I_0}^i(R_0))) \\ &\overset{(3)}{=} \mathrm{Hom}_{R_0}(R, \mathrm{H}_{I_0}^i(R_0) \otimes_{R_0} D_{R_0}(\mathrm{H}_{I_0}^i(R_0))) \\ &= \mathrm{Hom}_{R_0}(R, \mathrm{E}_{R_0}(k)) \\ &\overset{(2)}{=} \mathrm{E}_R(k) \end{aligned}$$

For (3) we use the fact that R is a finite-rank free R_0-module.

7.4.2 Theorem

Let R be a noetherian local complete regular ring of equicharacteristic zero, $I \subseteq R$ an ideal of height $h \geq 1$, $x_1, \ldots, x_h \in I$ an R-regular sequence and assume that

$$\mathrm{H}_I^l(R) = 0 \text{ for every } l > h \quad .$$

Then $\mathrm{H}_I^h(D(\mathrm{H}_I^h(R)))$ is either $\mathrm{E}_R(k)$ or zero.
Proof:
We set

$$D := D(\mathrm{H}_{(x_1, \ldots, x_h)R}^h(R))$$

By theorem 1.1.2, we know that $x_1, \ldots, x_h$ is a D-regular sequence and, therefore, we have

$$\mathrm{H}^0_{(x_1,\ldots,x_h)R}(D) = \ldots = \mathrm{H}^{h-1}_{(x_1,\ldots,x_h)R}(D) = 0 \quad .$$

Because of this, an easy spectral sequence argument (applied to the composed functor $\Gamma_I \circ \Gamma_{(x_1,\ldots,x_h)R}$ and to the R-module D) shows that

$$\mathrm{H}^h_I(D) = \Gamma_I(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(D)) \subseteq \mathrm{H}^h_{(x_1,\ldots,x_h)R}(D) = \mathrm{E}_R(k) \quad .$$

The last equality is theorem 7.4.1. But, from subsection 7.2 and from [Ly1, Example 2.1 (iv)], it is clear that $\mathrm{H}^h_I(D)$ has a D-module structure and so, from [Ly1, Theorem 2.4 (b)], we deduce that $\mathrm{H}^h_I(D)$ is either $\mathrm{E}_R(k)$ or zero. Furthermore, the natural injection

$$\mathrm{H}^h_I(R) \subseteq \mathrm{H}^h_{(x_1,\ldots,x_h)R}(R)$$

induces a surjection

$$D \to D(\mathrm{H}^h_I(R))$$

and hence, as H^h_I is right-exact, a surjection

$$\mathrm{H}^h_I(D) \to \mathrm{H}^h_I(D(\mathrm{H}^h_I(R))) \quad .$$

But again, the last module has a D-module structure, and thus, from [Ly1, Theorem 2.4 (b)] and from what we know already, we conclude the statement.

8 Attached prime ideals and local homology

8.1 Attached prime ideals – basics

This subsection is a collection of definitions and facts about primary and secondary representation, both in general situations (i. e. we do not always assume that our modules have any finiteness properties). We will make use of these facts in subsection 8.2. [BS] is a reference for the notion of attached primes (of local cohomology modules).

8.1.1 Definition and remark

Let R be a ring, $M \neq 0$ an R-module and N an R-submodule of M. M is coprimary iff the following condition holds: For every $x \in R$ the endomorphism $M \xrightarrow{x} M$ given by multiplication by x is injective or nilpotent (i. e. $\exists N \in \mathbb{N} : x^N \cdot M = 0$, note that for finitely generated M this is equivalent to $\forall_{m \in M} \exists N \in \mathbb{N} : x^N \cdot m = 0$). If M is coprimary $\sqrt{\operatorname{Ann}_R(M)}$ is a prime ideal of R. In general we say N is a primary submodule of M iff M/N is coprimary. Now let $U_1, \ldots, U_s \subseteq M$ be submodules of M. We say the s-tuple $(U_1, \ldots, U_s)$ is a primary decomposition of (the zero ideal of) M iff the following two conditions hold:
(i) $U_1 \cap \ldots \cap U_s = 0$.
(ii) All U_i are primary submodules of M.

In this case $(U_1, \ldots, U_s)$ is called a minimal primary decomposition of M iff, in addition, the following two statements hold:
(iii) Every $U_1 \cap \ldots \cap \hat{U}_i \cap \ldots \cap U_s$ is not zero.
(iv) The ideals $\sqrt{\operatorname{Ann}_R(M/U_i)}$ (for $i = 1, \ldots, s$) are pairwise different.

It is clear that if there exists a primary decomposition of M there is also a minimal one.

8.1.2 Definition and remark

Let R be a noetherian ring, M an R-module and assume there exists a minimal primary decomposition $(U_1, \ldots, U_s)$ of M. Then the set

$$\{\sqrt{\operatorname{Ann}_R(M/U_i)} | i = 1, \ldots, s\} =: \operatorname{Ass}_R(M)$$

does not depend on the choice of a minimal primary decomposition of M (the proof of this goes just like the well-known proof in case M is finite). We say the prime ideals of $\operatorname{Ass}_R(M)$ are associated to M

8.1.3 Remark

Let R be a noetherian ring and M a noetherian (i. e. finitely generated) R-module. Then it is well-known that M has a (minimal) primary decomposition. Note that this holds without the hypothesis R is noetherian, but anyway M being noetherian implies that $R/\operatorname{Ann}_R(M) =: \overline{R}$ is noetherian and M is a $\overline{R}$-module.

8.1.4 Definition

Let R be a noetherian ring and M an R-module. One defines

$$\operatorname{Ass}_R(M) := \{\mathfrak{p} \subseteq R \text{ prime ideal } | \exists m \in M : \mathfrak{p} = \operatorname{Ann}_R(m)\}.$$

It is easy to see that this definition agrees with the above one whenever M has a primary decomposition.

8.1.5 Definition and remark

Let R be a ring and $M \neq 0$ an R-module. By definition, M is secondary iff for every $x \in R$ the endomorphism $M \xrightarrow{x} M$ given by multiplication by x is either surjective or nilpotent. Now let M be arbitrary and $U_1, \ldots, U_s \subseteq M$ R-submodules. We say the s-tuple $(U_1, \ldots, U_s)$ is a secondary decomposition of M iff the following two conditions hold: $U_1 + \ldots + U_s = M$ and all U_i are secondary. In this case the secondary decomposition $(U_1, \ldots, U_s)$ is called minimal iff the following two conditions hold: All $U_1 + \ldots + \hat{U}_i + \ldots + U_s$ are proper subsets of M and all $\sqrt{\mathrm{Ann}_R(U_i)}$ are pairwise different. Again, existence of a secondary decomposition implies existence of a minimal one.

8.1.6 Definition and remark

Let R be a noetherian ring and M an R-module; assume there exists a minimal secondary decomposition $(U_1, \ldots, U_s)$ of M. Then the set

$$\mathrm{Att}_R(M) := \{\sqrt{\mathrm{Ann}_R(U_i)} | i = 1, \ldots s\}$$

does not depend on the choice of a minimal secondary decomposition of M. We say the prime ideals in $\mathrm{Att}_R(M)$ are attached to M.

8.1.7 Remark

Let R be a noetherian ring and M an artinian R-module. Then there exists a (minimal) secondary decomposition of M. The proof is simply a dual version of the proof of 8.1.3 (which is, of course, well-known). Again this works also if R is not noetherian.

8.1.8 Definition

Let R be a noetherian ring and M an R-module. We define

$$\mathrm{Att}_R(M) := \{\mathfrak{p} \subseteq R \text{ prime ideal } | \exists \text{ an } R\text{-submodule } U \subseteq M : \mathfrak{p} = \mathrm{Ann}_R(M/U)\}.$$

Is is not very difficult to see that this definition agrees with the first one if M has a secondary decomposition.

8.1.9 Remark

Let $(R, \mathfrak{m})$ be a noetherian local ring, M an R-module and $(U_1, \ldots, U_s)$ a minimal primary decomposition of M. The following implications are clear by duality:

(i) $U_1 \cap \ldots \cap U_s = 0 \Rightarrow D(M/U_1) + \ldots + \ldots D(M/U_s) = D(M)$

(ii) M/U_i is coprimary $\Rightarrow D(M/U_i)$ is secondary (for every i)

(iii) The primary decomposition $(U_1, \ldots, U_s)$ of M is minimal $\Rightarrow$
the secondary decomposition $(D(M/U_1), \ldots, D(M/U_s))$ of $D(M)$ is minimal.

(iv) $\mathrm{Ann}_R(M/U_i) = \mathrm{Ann}_R(D(M/U_i))$ (for every i)

Thus we have

$$\mathrm{Ass}_R(M) = \mathrm{Att}_R(D(M)) \quad .$$

In a very similar way the following statement holds: Any (minimal) secondary decomposition of M induces a (minimal) primary decomposition of $D(M)$. In particular, if M has a secondary decomposition:

$$\mathrm{Att}_R(M) = \mathrm{Ass}_R(D(M)) \quad .$$

8.1.10 Remark
It is true that if $U_1, \dots, U_s$ are arbitrary submodules of R such that $(D(M/U_1), \dots, D(M/Us))$ is a (minimal) secondary decomposition of $D(M)$ then $(U_1, \dots, U_s)$ is a (minimal) primary decomposition of M, but note that we do not know that every submodule of $D(M)$ is of the form $D(M/U)$ for some submodule U of M. Similarly, if $U_1, \dots, U_s$ are arbitrary submodules of M such that $(D(M/U_1), \dots, D(M/U_s))$ is a (minimal) primary decomposition of $D(M)$ then $(U_1, \dots, U_s)$ is a (minimal) secondary decomposition of M.

8.1.11 Remark
Let $(R, \mathfrak{m})$ be a noetherian local ring, $\mathfrak{p}$ a prime ideal of R and M an R-module. Then

$$\begin{aligned} \mathfrak{p} \in \operatorname{Ass}_R(M) &\iff \exists \text{ finitely generated submodule } U \text{ of } M : \mathfrak{p} = \operatorname{Ann}_R(U), \\ \mathfrak{p} \in \operatorname{Att}_R(D(M)) &\iff \exists \text{ submodule } U' \text{ of } D(M) : \mathfrak{p} = \operatorname{Ann}_R(D(M)/U'). \end{aligned}$$

In particular the existence of a submodule U of M satisfying $\mathfrak{p} = \operatorname{Ann}_R(U)$ implies $\mathfrak{p} \in \operatorname{Att}_R(D(M))$. Therefore we have

$$\operatorname{Ass}_R(M) \subseteq \operatorname{Att}_R(D(M)).$$

This inclusion is strict in general: Take for example $M = \mathrm{E} = \mathrm{E}_R(R/\mathfrak{m})$, an R-injective hull of $R/\mathfrak{m}$: $\operatorname{Ass}_R(\mathrm{E}_R(R/\mathfrak{m})) = \{\mathfrak{m}\}$, but $D(\mathrm{E}) = \hat{R}$ and so $\operatorname{Att}_R(D(\mathrm{E})) = \operatorname{Spec}(R)$. But nevertheless a stronger inclusion holds (plug in $D(M)$ for M in theorem 8.1.12 to see that it is actually stronger):

8.1.12 Theorem
Let $(R, \mathfrak{m})$ be a noetherian local ring and M an R-module. Then

$$\operatorname{Ass}_R(D(M)) \subseteq \operatorname{Att}_R(M)$$

and the sets of prime ideals maximal in each side respectively coincide:

$$\{\mathfrak{p} | \mathfrak{p} \text{ maximal in } \operatorname{Ass}_R(D(M))\} = \{\mathfrak{p} | \mathfrak{p} \text{ maximal in } \operatorname{Att}_R(M)\}.$$

Proof:
Let $\mathfrak{p} \in \operatorname{Ass}_R(D(M))$ be arbitrary. There exists a submodule U' of $D(M)$ such that $U' = R \cdot u' \cong R/\mathfrak{p}$ for some $u' \in U' \subseteq D(M)$. u' induces a monomorphism $\overline{u'} : M/\ker(u') \to \mathrm{E}$ and so we have

$$\mathfrak{p} = \operatorname{Ann}_R(U') = \operatorname{Ann}_R(u') = \operatorname{Ann}_R(\overline{u'}) = \operatorname{Ann}_R(M/\ker(u'));$$

this implies $\mathfrak{p} \in \operatorname{Att}_R(M)$. Having proved this we only have to show that an arbitrary prime ideal $\mathfrak{p}$ of R which is maximal in $\operatorname{Att}_R(M)$ is associated to $D(M)$: $\mathfrak{p} \in \operatorname{Att}_R(M)$ implies $M/\mathfrak{p}M \neq 0$ and so we must have $\operatorname{Hom}_R(R/\mathfrak{p}, D(M)) = D(M/\mathfrak{p}M) \neq 0$; but by the maximality hypothesis on $\mathfrak{p}$ implies $\mathfrak{p} \in \operatorname{Ass}_R(D(M))$.

8.1.13 Theorem
Let $(R, \mathfrak{m})$ be a noetherian local ring and M an R-module. Assume $(\mathfrak{p}_i)_{i \in \mathrm{N}}$ is a sequence of prime ideals attached to M; assume furthermore that $\mathfrak{q} := \bigcap_{i \in \mathrm{N}} \mathfrak{p}_i$ is a prime ideal of R. Then $\mathfrak{q}$ is also attached to M.
Proof:
For every i we choose a quotient M_i of M such that $\operatorname{Ann}_R(M_i) = \mathfrak{q}_i$. Now the canonically induced map $\iota : M \to \prod_{i \in \mathrm{N}} M_i$ induces a surjection $M \to \operatorname{im}(\iota)$; we obviously have $\bigcap_{i \in \mathrm{N}} \mathfrak{p}_i \subseteq \operatorname{Ann}_R(\operatorname{im}(\iota))$; on the other

hand, for every i and every $s \in R \backslash \mathfrak{p}_i$ there is a $\overline{m_i} \in M_i$ coming from an element $m_i \in M$ that has $s \cdot \overline{m_i} \neq 0$. But this implies that s cannot annihilate $\mathrm{im}(\iota)$; therefore

$$\mathrm{Ann}_R(\mathrm{im}(\iota)) = \bigcap_{i \in \mathbb{N}} \mathfrak{p}_i = \mathfrak{q}$$

and the statement follows.

8.2 Attached prime ideals – results

This subsection contains results on attached prime ideals (of local cohomology modules). Our technique bases on subsection 8.1 where some relations between attached primes of a module and associated primes of the Matlis dual of the same module were established. This method does not only lead to an easy proof of a known result (theorem 8.2.1, see also remark 8.2.2), but also enables us to find more attached prime ideals (of a local cohomology module, see theorem 8.2.3 and corollary 8.2.4 for details). Furthermore, the study of attached prime ideals leads to new evidence for conjecture (*) (this evidence comes, essentially, from theorem 8.1.13 which describes a property of the set of attached prime ideals that is necessary for being closed under generalization).

There are some results on the set of attached primes of local cohomology modules: In [MS, theorem 2.2] it was shown that if $(R, \mathfrak{m})$ is a noetherian local ring and M is a finitely generated R-module then

$$\mathrm{Att}_R(\mathrm{H}^{\dim(M)}_{\mathfrak{m}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) | \dim(R/\mathfrak{p}) = \dim(M)\}$$

holds. In [DY, Theorem A] this was generalized to

$$\mathrm{Att}_R(\mathrm{H}^{\dim(M)}_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) | \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = \dim(M)\},$$

where $\mathfrak{a} \subseteq R$ is an ideal and $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) := \max\{l \in \mathbb{N} | \mathrm{H}^l_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0\}$. We are going to show (theorem 8.2.1) that the results of section 8.1 lead to a natural proof of this result and, furthermore, to new results on the attached primes of local cohomology modules (8.2.3 – 8.2.6).

Let $(R, \mathfrak{m})$ be a noetherian local n-dimensional ring and $\mathfrak{a} \subseteq R$ an ideal. Then $\mathrm{H}^n_{\mathfrak{a}}(R)$ is an artinian R-module and hence

$$\mathrm{Ass}_R(D(\mathrm{H}^n_{\mathfrak{a}}(R))) = \mathrm{Att}_R(\mathrm{H}^n_{\mathfrak{a}}(R)).$$

Now assume that we have ($\mathrm{H}^n_{\mathfrak{a}}(R) \neq 0$ and) $\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^n_{\mathfrak{a}}(R))$; then we get

$$0 \neq \mathrm{H}^n_{\mathfrak{a}}(R)/\mathfrak{p}\,\mathrm{H}^n_{\mathfrak{a}}(R) = \mathrm{H}^n_{\mathfrak{a}}(R/\mathfrak{p}),$$

i. e. $\mathfrak{p} \in \mathrm{Assh}(R)(:= \{\mathfrak{q} \in \mathrm{Spec}(R) | \dim(R/\mathfrak{q}) = \dim(R)\})$ and $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n$.
Now suppose conversely that we have a prime ideal $\mathfrak{p}$ of R such that $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n$, equivalently $\mathrm{H}^n_{\mathfrak{a}}(\widehat{R/\mathfrak{p}}) \neq 0$. By Hartshorne-Lichtenbaum vanishing we get a prime ideal $\mathfrak{q} \subseteq \hat{R}$ satisfying $\mathfrak{p} = \mathfrak{q} \cap R$ and $\sqrt{\mathfrak{a}\hat{R} + \mathfrak{q}} = \mathfrak{m}_{\hat{R}}$(:=maximal ideal of $\hat{R}$); this in turn implies

$$0 \neq \mathrm{H}^n_{\mathfrak{a}\hat{R}}(\hat{R}/\mathfrak{q}) = \mathrm{H}^n_{\mathfrak{m}_{\hat{R}}}(\hat{R}/\mathfrak{q}).$$

Matlis duality theory shows that $\mathfrak{q} \in \operatorname{Ass}_{\hat{R}}(D(\mathrm{H}^n_{\mathfrak{a}\hat{R}}(\hat{R})))$. It is easy to see that

$$D(\mathrm{H}^n_{\mathfrak{a}\hat{R}}(\hat{R})) = D(\mathrm{H}^n_{\mathfrak{a}}(R)),$$

holds canonically, the D-functors taken over $\hat{R}$ resp. over R. Thus we have shown

$$\operatorname{Att}_R(\mathrm{H}^n_{\mathfrak{a}}(R)) = \{\mathfrak{p} | \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = n\}.$$

For every finitely generated R-module M we can apply this result to the ring $R/\operatorname{Ann}_R(M)$ and we get

8.2.1 Theorem
Let $(R, \mathfrak{m})$ be a noetherian local ring, $\mathfrak{a} \subseteq R$ ein Ideal and M a finitely generated n-dimensional R-module. Then

$$\operatorname{Att}_R(\mathrm{H}^n_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) | \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = n\}$$

holds.

8.2.2 Remark
This is [DY, Theorem A], where it was proved by different means.

In subsection 8.1 we established several relations between attached primes of a module and associated primes of the Matlis dual of the same module; theorem 8.2.3 is a consequence of these relations; we can retrieve more information from these relations to get new theorems on the attached primes of top local cohomology modules (remarks 8.2.5):

8.2.3 Theorem
Let $(R, \mathfrak{m})$ be a d-dimensional noetherian local ring.
(i) If J is an ideal of R such that $\dim(R/J) = 1$ and $\mathrm{H}^d_J(R) = 0$ then

$$\operatorname{Assh}(R) \subseteq \operatorname{Att}_R(\mathrm{H}^{d-1}_J(R))$$

holds. If, in addition, R is complete, one has

$$\operatorname{Att}_R(\mathrm{H}^{d-1}_J(R)) = \{\mathfrak{p} \in \operatorname{Spec}(R) | \dim(R/\mathfrak{p}) = d-1, \sqrt{\mathfrak{p}+J} = \mathfrak{m}\} \cup \operatorname{Assh}(R).$$

(ii) For any $x_1, \ldots, x_i \in R$ there is an inclusion

$$\{\mathfrak{p} \in \operatorname{Spec}(R) | x_1, \ldots, x_i \text{ is a part of a system of parameters of } R/\mathfrak{p}\} \subseteq \operatorname{Att}_R(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)).$$

Proof:
(i) Note that theorems 3.2.6 and 3.2.7 show that one has $\operatorname{Assh}(R) = \operatorname{Assh}(D(\mathrm{H}^{d-1}_J(R)))$ in the given situation and, if R is complete,

$$\operatorname{Ass}_R(D(\mathrm{H}^{d-1}_J(R))) = \{\mathfrak{p} \in \operatorname{Spec}(R) | \dim(R/\mathfrak{p}) = 1, \dim(R/(\mathfrak{p}+J)) = 0\} \cup \operatorname{Assh}(R) \quad .$$

Now we use theorem 8.1.12 and remark: If R is complete, given an arbitrary $\mathfrak{p} \in \operatorname{Att}_R(\mathrm{H}^{d-1}_J(R))$ it follows that $\mathrm{H}^{d-1}_J(R/\mathfrak{p}) \neq 0$ and hence, by Hartshorne Lichtenbaum vanishing, that $\dim(R/\mathfrak{p}) \geq d-1$ and, if $\dim(R/\mathfrak{p}) = d-1$, that $\mathfrak{p}+J$ is $\mathfrak{m}$-primary.

(ii) Follows from theorems 8.1.12 and 3.1.3 (ii).

8.2.4 Corollary
Let $(R, \mathfrak{m})$ be a noetherian local ring. For every $x \in R$ one has

$$\operatorname{Att}_R(\mathrm{H}^1_{xR}(R)) = \operatorname{Spec}(R) \setminus \mathfrak{V}(x).$$

Proof:
"$\subseteq$" Let $\mathfrak{p} \in \operatorname{Att}_R(\mathrm{H}^1_{xR}(R))$. Then

$$0 \neq \mathrm{H}^1_{xR}(R)/\mathfrak{p}\,\mathrm{H}^1_{xR}(R) = \mathrm{H}^1_{xR}(R/\mathfrak{p}) \Rightarrow x \notin \mathfrak{p} \quad .$$

"$\supseteq$" follows e. g. from 3.1.3 (ii).

8.2.5 Remarks
(i) It was shown in remark 1.2.1 that, for any $x_1, \ldots, x_i \in R$, there is an inclusion

$$\operatorname{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) |\, \mathrm{H}^i_{(x_1,\ldots,x_i)R}(R/\mathfrak{p}) \neq 0\}.$$

By what we have proved so far it is clear that there is a chain of inclusions

$$\operatorname{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))) \subseteq \operatorname{Att}_R(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) |\, \mathrm{H}^i_{(x_1,\ldots,x_i)R}(R/\mathfrak{p}) \neq 0\}.$$

(ii) Conjecture (*) says that, for any sequence $x_1, \ldots, x_i$ in R, the inclusion

$$\operatorname{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) |\, \mathrm{H}^i_{(x_1,\ldots,x_i)R}(R/\mathfrak{p}) \neq 0\}$$

is an equality; if this could be shown to be true, we could conclude that

$$\operatorname{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R})) = \operatorname{Att}_R(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)) \quad .$$

(iii) In the situation of theorem 8.2.3 (i) the attached primes of the top local cohomology module coincide with the associated primes of the Matlis dual of the top local cohomology module.

8.2.6 Remarks
We now assume that k is a field and $R = k[[X_1, \ldots, X_n]]$ is a power series algebra over k in n variables $X_1, \ldots, X_n$; let $i \in \{1, \ldots, n\}$. Theorems 4.2.1, 4.3.4 and 8.1.12 imply the following statements:
(i) In the case $i = n$ we have

$$\operatorname{Att}_R(\mathrm{H}^n_{(X_1,\ldots,X_n)R}(R)) = \{0\} \quad .$$

(ii) If $i = n - 1$,

$$\operatorname{Att}_R(\mathrm{H}^{n-1}_{(X_1,\ldots,X_{n-1})}(R) = \{0\} \cup \{pR | p \in R \text{ prime element, } p \notin (X_1, \ldots, X_{n-1})R\}$$

holds.
(iii) Finally we concentrate on the case $i = n - 2$, where we have the following statements:
(α) $\{0\} \in \operatorname{Att}_R(\mathrm{H}^{n-2}_{(X_1,\ldots,X_{n-2})R}(R))$;

(β) If $\mathfrak{p}$ is a height-two prime ideal of R such that $\sqrt{(X_1,\ldots,X_{n-2})R+\mathfrak{p}} = \mathfrak{m}$ then

$$\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^{n-2}_{(X_1,\ldots,X_{n-2})R}(R))$$

holds.
(γ) Conversely, $\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^{n-2}_{(X_1,\ldots,X_{n-2})R}(R))$ implies that $\mathrm{height}(\mathfrak{p}) \leq 2$;
(δ) If $p \in R$ is a prime element such that $p \notin (X_1,\ldots,X_{n-2})R$ then

$$pR \in \mathrm{Att}_R(\mathrm{H}^{n-2}_{(X_1,\ldots,X_{n-2})R}(R))$$

holds.
(ϵ) If $p \in R$ is a minimal generator of $(X_1,\ldots,X_{n-2})R$ then

$$pR \notin \mathrm{Att}_R(\mathrm{H}^{n-2}_{(X_1,\ldots,X_{n-2})R}(R))$$

holds.
(ζ) Because of theorems 4.3.4 and 8.1.12, for every prime element $p \in (X_1,\ldots,X_{n-2})R \cap (X_{n-1},X_n)R$ there exist infinitely many (pairwise different) prime ideals $(\mathfrak{p}_l)_{l\in\mathrm{N}}$ of height two attached to $\mathrm{H}^{n-2}_{(X_1,\ldots,X_{n-2})R}(R)$ and containing p. As any $q \in \bigcap_{l\in\mathrm{N}} \mathfrak{p}_l$ must satisfy $\mathrm{height}(p,q)R < 2$ it is clear that we have $pR = \bigcap_{l\in\mathrm{N}} \mathfrak{p}_l$. Now theorem 8.1.13 implies $pR \in \mathrm{Att}_R(\mathrm{H}^{n-2}_{(X_1,\ldots,X_{n-2})R}(R))$. But in view of theorems 1.2.3 and 8.1.12 it is clear that $pR \in \mathrm{Att}_R(\mathrm{H}^{n-2}_{(X_1,\ldots,X_{n-2})R}(R)$ is a necessary condition for conjecture (*). This gives new evidence for conjecture (*).

8.3 Local homology and a necessary condition for Cohen-Macaulayness

Let $(R,\mathfrak{m})$ be a noetherian, local ring, M an R-module and I an ideal of R. It is well-known that $\mathrm{H}^{\dim(M)}_I(M)$ is artinian for any proper ideal I of R provided M is finitely generated as R-module (cp. [Me]).
There is a theory of local homology modules (cp. [T1] and [T2]): If X is an artinian R-module and $\underline{x} = x_1,\ldots,x_r$ is a sequence of elements in $\mathfrak{m}$, the i-th local homology module $\mathrm{H}^{\underline{x}}_i(X)$ of X with respect to $\underline{x}$ is defined by

$$\varprojlim_{n\in\mathbf{N}} H_i(K_\bullet(x_1^n,\ldots,x_r^n;X)) \quad ,$$

where $K_\bullet(x_1^n,\ldots,x_r^n;X)$ is the Koszul complex of X with respect to $x_1^n,\ldots,x_r^n$ and H_i means taking the homology of this complex at the i-th position; then $\mathrm{H}^{\underline{x}}_i(\ )$ is an R-linear, covariant functor from the category of artinian R-modules to the category of R-modules.
We repeat the notions of Noetherian dimension $\mathrm{N.dim}(X)$ and width of X, $\mathrm{width}(X)$: For $X = 0$ one puts $\mathrm{N.dim}(X) = -1$, for $X \neq 0$ $\mathrm{N.dim}(X)$ denotes the least integer r such that $0 :_X (x_1,\ldots,x_r)R$ has finite length for some $x_1,\ldots,x_r \in \mathfrak{m}$. Now let $x_1,\ldots,x_n \in \mathfrak{m}$. $x_1,\ldots,x_n$ is an X-coregular sequence if

$$0 :_X (x_1,\ldots,x_{i-1})R \xrightarrow{x_i} 0 :_X (x_1,\ldots,x_{i-1})R$$

is surjective for $i = 1,\ldots,n$. $\mathrm{width}(X)$ is defined as the length of a (in fact any) maximal X-coregular sequence in $\mathfrak{m}$. Details on $\mathrm{N.dim}(X)$ and $\mathrm{width}(X)$ can be found in [Oo] and [Ro], here we cite one general fact: For any artinian R-module X

$$\mathrm{width}(X) \leq \mathrm{N.dim}(X) < \infty$$

holds and X is co-Cohen-Macaulay if and only if $\text{width}(X) = \text{N.dim}(X)$ holds (by definition). Tang has shown ([T1, Proposition 2.6]) that $\text{H}_{\mathfrak{m}}^{\dim(M)}(M)$ is co-Cohen-Macaulay (of Noetherian dimension $\dim(M)$) if M is a finitely generated Cohen-Macaulay R-module and ([T1, Theorem 3.1]) that

$$\text{H}_{\dim(M)}^{x_1,\ldots,x_d}(\text{H}_{\mathfrak{m}}^{\dim(M)}(M)) = \hat{M}$$

holds (here $x_1, \ldots, x_d$ is a s. o. p. of M and we still assume that M is Cohen-Macaulay). Tang asks ([T1, Remark 3.5]) if one can show that $\text{H}_d^{\underline{x}}(X)$ is finitely generated if X is an artinian R-module of N.dimension d and $\underline{x} = x_1, \ldots, x_d$ is such that $0 :_X \underline{x}$ has finite length.
In the example 8.3.1 below we give a negative answer to this question. However, under the additional assumption that R is complete, we show that $\text{H}_d^{\underline{x}}(X)$ is a finitely generated R-module (theorem 8.3.3) and draw some consequences establishing various duality results (theorem 8.3.5). As an application we present a necessary condition for a given finite R-module M to be Cohen-Macaulay (corollary 8.3.6).
$\underline{x}$ will always stand for a sequence $x_1, \ldots, x_d$ in $\mathfrak{m}$. The results of this and the next subsection can also be found in [H5].

8.3.1 Example
Let k be a field, T a variable and R the noetherian, local ring $k[T]_{(T)}$. Set $X := T^{-1} \cdot k[T^{-1}] := \{a_{-1}T^{-1} + \ldots + a_{-n}T^{-n} | n \in \mathbb{N}^+, a_{-1}, \ldots, a_{-n} \in k\}$. X has a $\hat{R} = k[[T]]$-structure (such that $T^m \cdot T^{-n} = T^{m-n}$ if $m - n \leq -1$ and $= 0$ if $m - n \geq 0$, where $m \geq 0, n \geq -1$) and thus it also has an R-module-structure. Every non-trivial R-submodule of X has the form $< T^{-n} >_X$ for some $n \geq 1$ and therefore X is an artinian R-module. Furthermore $(0 :_X T) = k \cdot T^{-1}$ is of finite length (and so $\text{N.dim}(X) = 1$) and $\text{H}_1^T(X)$ is the indirect limit over all $(0 :_X T^l)$, where the transition maps $(0 :_X T^{l+1}) \to (0 :_X T^l)$ are induced by multiplication by T. An elementary calculation shows $\text{H}_1^T(X) = k[[T]] = \hat{R}$ which is not finite as an R-module.

But more can be said:

8.3.2 Remark
Let $(R, \mathfrak{m})$ be a local noetherian regular d-dimensional ring, X an artinian co-Cohen-Macaulay R-module, $\text{N.dim}(X) = d$, $\underline{x} = x_1, \ldots, x_d \in \mathfrak{m}$ such that $(0 :_X \underline{x})$ is of finite length. Then

$$\text{H}_d^{\underline{x}}(X) \text{ is a finite } R\text{-module} \iff R \text{ is complete}$$

holds.
Proof:
$\Leftarrow$ follows from theorem 8.3.3 below. $\Rightarrow$: From [T1, Remark 3.5] it follows that $\text{depth}(\text{H}_d^{\underline{x}}(X)) = d$ both as an $R-$ and as an $\hat{R}$-module; but now the Auslander-Buchsbaum formula implies that $\text{H}_d^{\underline{x}}(X)$ is a finite free $\hat{R}$-module and so we must have $\hat{R} = R$.

From now on we assume that R is complete and show at first that the top local homology module is always finite; this is done, essentially, by Matlis duality.

8.3.3 Theorem
Let $(R, \mathfrak{m})$ be a noetherian, local, complete ring, X an artinian R-module of N.dimension d; let $x_1, \ldots, x_d \in \mathfrak{m}$ be such that $0 :_X (x_1, \ldots, x_d)R$ has finite length. Then $\text{H}_d^{\underline{x}}(X)$ is a finitely generated R-module.

Proof:
$x_1, \ldots, x_d$ form a system of parameters for $D(X)$, because

$$D(X)/(x_1, \ldots, x_d)D(X) = D(0 :_X (x_1, \ldots, x_d)R)$$

has finite length and $\dim(D(X)) = \mathrm{N.dim}(X) = d$. Using Matlis-duality we have

$$\begin{aligned} \mathrm{H}^{\underline{x}}_d(X) &= \mathrm{H}^{\underline{x}}_d(D(D(X))) \\ &= \varprojlim_{n \in \mathbf{N}} H_d(K_\bullet(x_1^n, \ldots, x_d^n; D(D(X)))) \\ &= \varprojlim_{n \in \mathbf{N}} D(H^d(K^\bullet(x_1^n, \ldots, x_d^n; D(X)))) \\ &= D(\varinjlim_{n \in \mathbf{N}} H^d(K^\bullet(x_1^n, \ldots, x_d^n; D(X)))) \\ &= D(\mathrm{H}^d_{(x_1, \ldots, x_d)R}(D(X))) \quad , \end{aligned}$$

and the last module is finitely generated because $\mathrm{H}^d_{(x_1, \ldots, x_d)R}(D(X))$ is artinian.

8.3.4 Corollary
Let $(R, \mathfrak{m})$ be a noetherian, local, complete ring and X a co-Cohen-Macaulay R-module of N.dimension d; let $x_1, \ldots, x_d \in \mathfrak{m}$ be such that $0 :_X (x_1, \ldots, x_d)R$ has finite length. Then $\mathrm{H}_d^{x_1, \ldots, x_d}(X)$ is a Cohen-Macaulay module. In particular if $d = \dim(R)$, $\mathrm{H}_d^{x_1, \ldots, x_d}(X)$ is a maximal Cohen-Macaulay module.
Proof:
The statements follow from theorem 8.3.3 and [T1, Remark 3.5].

Let $(R, \mathfrak{m})$ be a noetherian, local, complete ring. Let $\mathcal{N}$ (resp. $\mathcal{A}$) denote the set of isomorphism classes of noetherian (resp. of artinian) R-modules. We have maps F_1 and F_2 from $\mathcal{N}$ to $\mathcal{A}$ induced by

$$M \overset{F_1}{\mapsto} \text{Matlis dual of } M$$

and

$$M \overset{F_2}{\mapsto} \mathrm{H}_{\mathfrak{m}}^{\dim(M)}(M)$$

For F_2 it does not make any difference if we take $\mathrm{H}^{\dim(M)}_{(x_1, \ldots, x_{\dim(M)})R}(M)$ instead of $\mathrm{H}_{\mathfrak{m}}^{\dim(M)}(M)$ (for any system of parameters $x_1, \ldots, x_{\dim(M)}$ of M). Similarly we have maps G_1 and G_2 from $\mathcal{A}$ to $\mathcal{N}$ induced by

$$X \overset{G_1}{\mapsto} \text{Matlis-dual of } X$$

and

$$X \overset{G_2}{\mapsto} \mathrm{H}_{\mathrm{N.dim}(X)}^{x_1, \ldots, x_{\mathrm{N.dim}(X)}}(X)$$

(here $x_1, \ldots, x_{\mathrm{N.dim}(X)}$ are such that $0 :_X (x_1, \ldots, x_{\mathrm{N.dim}(X)})R$ has finite length). By Matlis-duality we have

$$F_1 \circ G_1 = \mathrm{id}_{\mathcal{A}}, G_1 \circ F_1 = \mathrm{id}_{\mathcal{N}} \quad .$$

From the proof of theorem 8.3.3 one understands that

$$F_1 \circ G_2 = F_2 \circ G_1 =: T$$

and hence

$$G_1 \circ F_2 = G_2 \circ F_1 =: T' \quad ,$$

$$G_2 = G_1 \circ F_2 \circ G_1 = G_1 \circ T, F_2 = F_1 \circ G_2 \circ F_1 = F_1 \circ T' \quad .$$

8.3.5 Theorem

Let $(R, \mathfrak{m})$ be a noetherian, local, complete ring. Let M be a noetherian and X an artinian R-module. Then
(i) If M is Cohen-Macaulay, then $F_2(M)$ is co-Cohen-Macaulay.
(ii) If M is Cohen-Macaulay, then $F_1(M)$ is co-Cohen-Macaulay.
(iii) If X is co-Cohen-Macaulay, then $G_2(M)$ is Cohen-Macaulay.
(iv) If X is co-Cohen-Macaulay, then $G_1(M)$ is Cohen-Macaulay.
Proof:
(ii) and (iv) are easily proved using Matlis-duality theory. (i) is proved by [T1, Proposition 2.6]) and now (iii) follows from $G_2 = G_1 \circ F_2 \circ G_1$.

Let $\mathcal{N}_0$ (resp. $\mathcal{A}_0$) denote the set of isomorphism classes of noetherian Cohen-Macaulay modules (resp. of artinian co-Cohen-Macaulay modules). Then, by theorem 8.3.5, F_1, F_2, G_1, G_2 induce maps between $\mathcal{N}_0$ and $\mathcal{A}_0$ in an obvious way. [T1, theorems 3.1 and 3.4] imply $F_2 \circ G_2 = \mathrm{id}_{\mathcal{A}_0}$ and $G_2 \circ F_2 = \mathrm{id}_{\mathcal{N}_0}$. We deduce $G_1 = G_2 \circ F_1 \circ G_2$, $F_1 = F_2 \circ G_1 \circ F_2, T^2 = \mathrm{id}, {T'}^2 = \mathrm{id}$ on $\mathcal{N}_0$ and $\mathcal{A}_0$.

As an application we get a necessary condition for a finite module to be Cohen-Macaulay:

8.3.6 Corollary

(i) Let ω_R be a dualizing module for R (it exists uniquely up to isomorphism since R is complete). Assume that M is Cohen-Macaulay. Then $\mathrm{Ext}_R^{\dim(R)-\dim(M)}(M, \omega_R)$ is Cohen-Macaulay.
(ii) In particular if there exists an ideal I of R such that $I \subseteq \mathrm{Ann}_R(M)$, $\dim(R/I) = \dim(M)$ and R/I is Gorenstein, Cohen-Macaulayness of M implies Cohen-Macaulayness of $\mathrm{Hom}_{\overline{R}}(M, \overline{R})$ (here $\overline{R} := R/I$). Such an ideal I exists, for example, if R itself is Gorenstein.
Proof:
The statements follow from local duality and theorem 8.3.5.

8.4 Local homology and Cohen-Macaulayfications

In the text following theorem 8.3.5 we have seen $G_2 \circ F_2 = \mathrm{id}_{\mathcal{N}_0}$ and $F_2 \circ G_2 = \mathrm{id}_{\mathcal{A}_0}$. Now we turn our interest to the question: What can be said about $G_2 \circ F_2$ in general, that is, on $\mathcal{N}$?

8.4.1 Definition

Let $(R, \mathfrak{m})$ be a noetherian, local, complete ring and M a noetherian (i. e. finitely generated) R-module. Let $\tilde{M}$ be a finitely generated R-module containing M as a submodule. We say $\tilde{M}$ is a Cohen-Macaulayfication of M if the following three conditions hold:
(i) $\tilde{M}$ is Cohen-Macaulay.
(ii) $\dim(\tilde{M}) = \dim(M)$.
(iii) $\dim(\tilde{M}/M) \leq \dim M - 2$ (this condition is equivalent to $\mathrm{H}_{\mathfrak{m}}^{\dim(M)-1}(\tilde{M}/M) = \mathrm{H}_{\mathfrak{m}}^{\dim(M)}(\tilde{M}/M) = 0$).

In the sequel we won't always strictly distinguish between a module M and its isomorphism class, for reasons of simplicity.

8.4.2 Theorem
Let $(R, \mathfrak{m})$ be a noetherian, local, complete ring and M a noetherian R-module. If M has a Cohen-Macaulayfication, it has (up to an M-isomorphism) only one Cohen-Macaulayfication, namely $(G_2 \circ F_2)(M)$.
Proof:
Let $\tilde{M}$ be a Cohen-Macaulayfication of M. We consider the short exact sequence $0 \to M \to \tilde{M} \to \tilde{M}/M \to 0$ and its long exact cohomology sequence induced by applying $\Gamma_{\mathfrak{m}}$ to it: Because of condition (iii) of definition 8.4.1 we get a canonical isomorphism

$$\mathrm{H}_{\mathfrak{m}}^{\dim(M)}(M) = \mathrm{H}_{\mathfrak{m}}^{\dim(M)}(\tilde{M}) \overset{8.4.1\ (ii)}{=} \mathrm{H}_{\mathfrak{m}}^{\dim(\tilde{M})}(\tilde{M})$$

and therefore $\tilde{M} = (G_2 \circ F_2)(\tilde{M}) = (G_2 \circ F_2)(M)$.

8.4.3 Remark
Goto (cp. [Go]) has shown: If $(A, \mathfrak{m})$ is a noetherian, local, d-dimensional ring with total quotient ring $Q(A)$, the following conditions are equivalent:
(i) There is a Cohen-Macaulay ring B between A and $Q(A)$ such that B is finitely generated as an A-module, $\dim(B_{\mathfrak{n}}) = d$ for every maximal ideal $\mathfrak{n}$ of B and $\mathfrak{m} \cdot B \subseteq A$.
(ii) A is a Buchsbaum ring (see [SV] for details on Buchsbaum rings) and $\mathrm{H}_{\mathfrak{m}}^i(A) = 0$ for $i \neq 1, d$.
In this case, if $d \geq 2$, B is uniquely determined and Goto ([Go]) calls it the Cohen-Macaulayfication of A.

8.4.4 Remark
Cohen-Macaulayfication in our sense is a generalization of Goto's concept of Cohen-Macaulayfication:

8.4.5 Theorem
Let $(R, \mathfrak{m})$ be a noetherian, local, complete ring, and assume that the Cohen-Macaulayfication B of R (in the sense of Goto) exists. Then B is also a Cohen-Macaulayfication in our sense.
Proof:
Because of $\mathfrak{m} \cdot B \subseteq R$ we have $\mathfrak{m} \cdot (B/R) = 0$, which implies that B/R is a finite-dimensional $R/\mathfrak{m}$-vector space. Because of $d = \dim(R) \geq 2$ we must have $\mathrm{H}_{\mathfrak{m}}^{d-1}(B/R) = \mathrm{H}_{\mathfrak{m}}^{d}(B/R) = 0$.

8.4.6 Remark
In particular if $(R, \mathfrak{m})$ is a noetherian, local, complete Buchsbaum-ring of dimension $d \geq 2$ such that $\mathrm{H}_{\mathfrak{m}}^i(R) = 0$ for $i \neq 1, d$, the R-module R has a Cohen-Macaulayfication.

8.4.7 Example
An easy example is given by $R = k[[x_1, x_2, x_3, x_4]]/(x_1, x_2) \cap (x_3, x_4)$. In the sense of Goto as well as in our sense R has a Cohen-Macaulayfication given by $(k[[x_1, \ldots, x_4]]/(x_1, x_2)) \oplus (k[[x_1, \ldots, x_4]]/(x_3, x_4))$; this can be seen either directly or by remarking that R is a 2-dimensional Buchsbaum ring with $\mathrm{H}_{\mathfrak{m}}^i(R) = 0$ for $i \neq 1, 2$.

References

[Ba] Bass, H. On the ubiquity of Gorenstein rings, *Math. Z.* **82**, (1963) 8–28.

[BHe] Brodmann, M. and Hellus, M. Cohomological patterns of coherent sheaves over projective schemes, *Journal of Pure and Applied Algebra* **172**, (2002) 165–182.

[BH] Bruns, W. and Herzog, J. Cohen-Macaulay Rings, *Cambridge University Press*, (1993).

[Bj] Bjork, J.-E. Rings of Differential Operators, *Amsterdam North-Holland*, (1979)

[BS] Brodmann, M. P. and Sharp, R. J. Local Cohomology, *Cambridge studies in advanced mathematics* **60**, (1998).

[CN] Cowsik, R. and Nori, M. Affine curves in characteristic p are set-theoretic complete intersections, *Invent. Math.* **45**, (1978) 111–114.

[DY] Dibaei, M. T. and Yassemi, S. Attached primes of the top local cohomology modules with respect to an ideal, *Arch. Math.* **84**, (2005) 292–297.

[Ei] Eisenbud, D. Commutative Algebra with A View Toward Algebraic Geometry, *Springer Verlag*, (1995).

[Go] Goto, S. On the Cohen-Macaulayfication of certain Buchsbaum rings, *Nagoya Math. J.* Vol. **80**, (1980) 107-116.

[Gr] Grothendieck, A. Local Cohomology, *Lecture Notes in Mathematics, Springer Verlag*, (1967).

[Ha1] Hartshorne, R. Affine Duality and Cofiniteness, *Invent. Math.* **9**, (1970) 145–164.

[Ha2] Hartshorne, R. Complete intersections in characteristic $p > 0$, *Amer. J. Math.* **101**, (1979) 380–383.

[H1] Hellus, M. On the set of associated primes of a local cohomology module, *J. Algebra* **237**, (2001) 406–419.

[H2] Hellus, M. On the associated primes of Matlis duals of top local cohomology modules, *Communications in Algebra* **33**, (2001), no. 11, 3997–4009.

[H3] Hellus, M. Matlis duals of top local cohomology modules and the arithmetic rank of an ideal, to appear in *Communications in Algebra*.

[H4] Hellus, M. Attached primes and Matlis duals of local cohomology modules, submitted to *Archiv der Mathematik*

[H5] Hellus, M. Local Homology, Cohen-Macaulayness and Cohen-Macaulayfications, to appear in *Algebra Colloquium*.

[H6] Hellus, M. Lokale Kohomologie, *Dissertation*, Regensburg (1999).

[HS1] Hellus, M. and Stückrad, J. Matlis duals of top Local Cohomology Modules, submitted to *Proceedings of the American Mathematical Society*.

[HS2] Hellus, M. and Stückrad, J. Generalization of an example of Hartshorne concerning local cohomology, preprint.

[Hu] Huneke, C. Problems on Local Cohomology, *Res. Notes Math.* **2**, (1992) 93–108.

[HL] Huneke, C. and Lyubeznik, G. On the vanishing of local cohomology modules, *Invent. Math.* **102**, (1990) 73-93.

[Ly1] Lyubeznik, G. Finiteness properties of local cohomology modules (an application of D-modules to Commutative Algebra), *Invent. Math.* **113**, (1993), 41–55.

[Ly2] Lyubeznik, G. A survey of problems and results on the number of defining equations, Commutative Algebra, *Math. Sci. Res. Inst. Publ.* **15**, Springer, (1989), 375–390.

[Ma] Matsumura, H. Commutative ring theory, *Cambridge University Press*, (1986).

[MD] MacDonald I. G. Secondary representation of modules over a commutative ring, *Symp. Math.* **XI**, (1973) 23–43.

[Me] Melkersson, L. Some applications of a criterion for artinianness of a module, *J. Pure and Appl. Alg.* **101**, (1995) 293-303.

[Ms] Matlis, E. Injective modules over Noetherian rings, *Pacific J. Math.* **8**, (1958) 511–528.

[MS] MacDonald I. G. and Sharp, R. Y. An elementary proof of the non-vanishing of certain local cohomology modules, *Quart. J. Math. Oxford* **23**, (1972) 197–204.

[MV] Marley, M. and Vassilev, J.C. Local cohomology modules with infinite dimensional socles, *Proc. Amer. Math. Soc.* Vol. **132**, No **12**, (2004) 3485–3490

[Og] Ogus, A., Local Cohomological Dimension of Algebraic Varieties, *Ann. Math.* Vol. **98**, (1973) 327–365

[Oo] Ooishi, A. Matlis duality and width of a module, *Hiroshima Math. J.* **6**, (1976) 573–587.

[Ro] Roberts R. N. Krull dimension for Artinian modules over quasi local commutative rings, *Quart. J. Math. (Oxford)(3)* **26**, (1975) 269–273.

[SS] Scheja, G. and Storch, U. Regular Sequences and Resultants, *AK Peters*, (2001).

[SV] Stückrad, J. and Vogel, W. Buchsbaum Rings and Applications. *VEB Deutscher Verlag der Wissenschaften, Berlin, Germany.*

[T1] Tang Z. M. Local Homology and Local Cohomology, *Algebra Colloquium* **11**:4, (2004) 467–476.

[T2] Tang Z. M. Local homology theory for Artinian modules, *Comm. Alg.* **22**, (1994) 2173–2204.

Deutsche Zusammenfassung

Lokale Kohomologie und Matlis-Dualität

Eine algebraische Menge X heißt (mengentheoretisch) vollständiger Durchschnitt, wenn sie von codim(X) vielen algebraischen Gleichungen "ausgeschnitten" werden kann (etwa in einem affinen oder projektiven Raum). Es ist bekannt, dass, im Falle positiver Charakteristik, jede Kurve im $n-$dimensionalen affinen Raum mengentheoretisch vollständiger Durchschnitt ist ([CN]). Auf der anderen Seite sind im Zusammenhang mit mengentheoretisch vollständigen Durchschnitten bemerkenswert viele Fragen unbeantwortet. Als Beispiele seien angeführt: Ist jeder abgeschlossene Punkt in $\mathbf{P}^2_{\mathbf{Q}}$ (zweidimensionaler projektiver Raum über den rationalen Zahlen) mengentheoretisch vollständiger Durchschnitt? Ist jede Kurve in $\mathbf{A}^3_{\mathbf{C}}$ (dreidimensionaler affiner Raum über den komplexen Zahlen) mengentheoretisch vollständiger Durchschnitt? Zu diesen und vielen weiteren verwandten Fragen enthält [Ly2] eine Übersicht.

Ein weiteres Beispiel ist die Kurve $C_4 \subseteq \mathbf{P}^3_k$, die durch

$$(u^4 : u^3v : uv^3 : v^4)$$

parametrisiert ist. Es ist, zumindest im Falle der Charakteristik Null, unbekannt, ob C_4 mengentheoretisch vollständiger Durchschnitt ist; eine offensichtliche Obstruktion wäre $\mathrm{H}^3_{I_{C_4}}(k[X_0, X_1, X_2, X_3]) \neq 0$ (wobei I_{C_4} das Verschwindungsideal von C_4 bezeichnet). Es ist aber bekannt, dass

$$\mathrm{H}^3_{I_{C_4}}(k[X_0, X_1, X_2, X_3]) = 0$$

ist. Es ist sogar so, dass das (Nicht-)Verschwinden von lokalen Kohomologien im Allgemeinen nicht die Minimalzahl algebraischer Gleichungen, die die gegebene algebraische Menge "ausschneiden", bestimmt. Algebraisch ausgedrückt, bedeutet dies, dass (für ein Ideal I) die Ungleichung

$$\mathrm{cd}(I) < \mathrm{ara}(I)$$

gelten kann (hier bezeichnen cd(I) die (lokale) kohomologische Dimension von I und

$$\mathrm{ara}(I) := \min\{l \in \mathbb{N} | \exists r_1, \ldots, r_l \in R : \sqrt{I} = \sqrt{(r_1, \ldots, r_l)R}\}$$

die Minimalzahl algebraischer Gleichungen, die die zu I gehörende algebraische Menge "ausschneiden"). Übrigens enthält 5.1 ein konkretes Beispiel für das Vorliegen dieser Ungleichung. Auf der anderen Seite enthalten die Matlis-Duale gewisser lokaler Kohomologiemoduln Informationen darüber, ob ein mengentheoretisch vollständiger Durchschnitt vorliegt oder nicht – dies ist der Inhalt von

1.1.4 Korollar
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler Ring, $I \subsetneq R$ ein echtes Ideal, $h \in \mathbb{N}$ und $\underline{f} = f_1, \ldots, f_h \in I$ eine R-reguläre Folge. Dann sind äquivalent:
(i) $\sqrt{\underline{f}R} = \sqrt{I}$ (d. h. I ist mengentheoretisch vollständiger Durchschnitt).
(ii) $\mathrm{H}^l_I(R) = 0$ für jedes $l > h$ und $\underline{f}$ ist eine $D(\mathrm{H}^h_I(R))$-quasi-reguläre Folge.
(ii) $\mathrm{H}^l_I(R) = 0$ für jedes $l > h$ und $\underline{f}$ ist eine $D(\mathrm{H}^h_I(R))$-reguläre Folge.

Dieses Ergebnis legt es nahe, Matlis-Duale von lokalen Kohomologiemoduln zu studieren, insbesondere ihre assoziierten Primideale; dies sind auch die Hauptziele dieser Arbeit. Die erhaltenen Ergebnisse und verwendeten Methoden führen auch zu verschiedenen Anwendungen, die in Kapitel 6 versammelt sind. Darüber hinaus ergeben sich Zusammenhänge zur (lokalen) Kohomologie formaler Schemata (7.1), zu sogenannten "attached" Primidealen von lokalen Kohomologiemoduln (8.1, 8.2) und zum Begriffe der lokalen Homologie (8.3, 8.4).

Folgende Bezeichnungen seien vereinbart: Sind R ein Ring, $I \subseteq R$ ein Ideal und M ein R-Modul, so bezeichnet $\mathrm{H}^l_I(M)$ die l-te lokale Kohomologie von M mit Träger in I; ist $(R, \mathfrak{m})$ ein lokaler Ring, so ist $\mathrm{E}_R(R/\mathfrak{m})$ eine (fixierte) R-injektive Hülle des R-Moduls $R/\mathfrak{m}$. Schließlich bezeichnet (über dem lokalen Ring $(R, \mathfrak{m})$) D den Matlis-Dualisierungsfunktor, d. h. $D(M) := \mathrm{Hom}_R(M, \mathrm{E}_R(R/\mathfrak{m}))$ für jeden R-Modul M. Zur Vermeidung von Missverständnissen werden wir gegebenenfalls D_R statt D schreiben.

Es folgt eine chronologische Übersicht des Inhalts der einzelnen Kapitel:

Ziel von Abschnitt 1.1 ist der Beweis des eingangs zitierten Korollars 1.1.4; dies geschieht, indem zunächst die folgenden Sätze 1.1.2 und 1.1.3 bewiesen werden, aus denen dann, im Wesentlichen durch Spezialisierung, Korollar 1.1.4 folgt:

1.1.2 Satz
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler Ring, $I \subseteq R$ ein Ideal, $h \geq 1$ und $\underline{f} = f_1, \ldots, f_h \in I$ eine Folge mit $\sqrt{\underline{f}R} = \sqrt{I}$ und so, dass

$$\mathrm{H}^{h-1-l}_I(R/(f_1, \ldots, f_l)R) = 0 \quad (l = 0, \ldots, h-3)$$

gilt (für $h \leq 2$ ist diese Bedingung leer). Dann ist $\underline{f}$ eine $D(\mathrm{H}^h_I(R))$-quasi-reguläre Folge.

1.1.3 Satz
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler Ring, $I \subseteq R$ ein Ideal, $h \geq 1$ und $\underline{f} = f_1, \ldots, f_h \in I$ so, dass

$$\mathrm{H}^l_I(R) = 0 \quad (l > h)$$

und

$$\mathrm{H}^{h-1-l}_I(R/(f_1, \ldots, f_h)R) = 0 \quad (l = 0, \ldots, h-2)$$

gelten (für $h \leq 1$ ist diese Bedingung leer) und so, dass $\underline{f}$ eine $D(\mathrm{H}^h_I(R))$-quasi-reguläre Folge ist. Dann gilt $\sqrt{I} = \sqrt{(f_1, \ldots, f_h)R}$.

Korollar 1.1.4 (siehe oben) legt es nahe, zu untersuchen, für welche $f \in R$ die Multiplikation mit x auf einem Matlis-Dual eines lokalen Kohomologiemoduls injektiv ist, mit anderen Worten, die Menge der Nullteiler auf einem solchen Modul zu bestimmen. Eine genauere Frage ist die nach der Menge der zu diesem Modul assoziierten Primideale. In diesem Zusammenhang verweisen wir auf

1.2.2 Vermutung
Sind $(R, \mathfrak{m})$ ein noetherscher lokaler Ring, $h > 0$ und $x_1, \ldots, x_h$ Elemente von R, so gilt

$$\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) = \{\mathfrak{p} \in \mathrm{Spec}(R) | \, \mathrm{H}^h_{(x_1,\ldots,x_h)R}(R/\mathfrak{p}) \neq 0\} \quad .$$

Diese Vermutung bezeichnen wir mit (*). Die Inklusion $\subseteq$ ist stets richtig, dies ist (unter anderem) der Inhalt von

1.2.1 Bemerkung
Sind $(R, \mathfrak{m})$ ein noetherscher lokaler Ring, $h > 0$ und $x_1, \ldots, x_h$ Elemente von R, so gilt

$$\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \mathrm{H}^h_{(x_1,\ldots,x_h)R}(R/\mathfrak{p}) \neq 0\} \quad .$$

Es gibt zu (*) äquivalente Aussagen:

1.2.3 Satz
Die folgenden Aussagen sind äquivalent:
(i) Vermutung (*) ist richtig, d. h. für jeden noetherschen lokalen Ring $(R, \mathfrak{m})$, jedes $h > 0$ und jede Folge $x_1, \ldots, x_h \in R$ gilt

$$\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) = \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \mathrm{H}^h_{(x_1,\ldots,x_h)R}(R/\mathfrak{p}) \neq 0\} \quad .$$

(ii) Für jeden noetherschen lokalen Ring $(R, \mathfrak{m})$, jedes $h > 0$ und jede Folge $x_1, \ldots, x_h \in R$ ist die Menge

$$Y := \mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)}(R)))$$

abgeschlossen unter Generalisierung, d. h. aus $\mathfrak{p}_0, \mathfrak{p}_1 \in \mathrm{Spec}(R), \mathfrak{p}_0 \subseteq \mathfrak{p}_1, \mathfrak{p}_1 \in Y$ folgt $\mathfrak{p}_0 \in Y$.
(iii) Für jeden noetherschen lokalen Integritätsring $(R, \mathfrak{m})$, jedes $h > 0$ und jede Folge $x_1, \ldots, x_h \in R$ gilt die Implikation

$$\mathrm{H}^h_{(x_1,\ldots,x_h)}(R) \neq 0 \Longrightarrow \{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))) \quad .$$

(iv) Für jeden noetherschen lokalen Ring $(R, \mathfrak{m})$, jeden endlich erzeugten R-Modul M, jedes $h > 0$ und jede Folge $x_1, \ldots, x_h \in R$ gilt die Gleichheit

$$\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(M))) = \{\mathfrak{p} \in \mathrm{Supp}_R(M) |\, \mathrm{H}^h_{(x_1,\ldots,x_h)R}(M/\mathfrak{p}M) \neq 0\} \quad .$$

Aussage (iv) ist also formal allgemeiner als Aussage (i), aber inhaltlich äquivalent dazu.

[HS1, Kapitel 0] enthält eine weitere Vermutung zur Struktur der Menge der assoziierten Primideale von $D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))$: Alle Primideale $\mathfrak{p}$, die maximal in $\mathrm{Ass}_R(D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R)))$ sind, haben die Dimension h: $\dim(R/\mathfrak{p}) = h$; diese Vermutung ist falsch, Bemerkung 1.2.4 enthält ein Gegenbeispiel (mit $\dim(R) - h = 2$).

Indem wir uns mit (quasi-)regulären Folgen auf Moduln der Form $D(\mathrm{H}^h_I(R))$ beschäftigen, stellt sich folgende Frage: Im allgemeinen ist $D(\mathrm{H}^h_I(R))$ nicht endlich erzeugt (viele Ergebnisse dieser Arbeit zeigen, dass dieser Modul im Allgemeinen unendlich viele assoziierte Primideale hat), der Begriff der regulären Folge auf nicht-endlichen Moduln lässt manche Eigenschaften vermissen: Beispielsweise gilt (über einem lokalen noetherschen Ring $(R, \mathfrak{m})$) für einen endlichen R-Modul M und eine M-reguläre Folge $r_1, \ldots, r_h \in R$, dass auch die Folge $r'_1, \ldots, r'_h \in R$ M-regulär ist, falls nur $(r_1, \ldots, r_h)R = (r'_1, \ldots, r'_h)R$ vorausgesetzt ist; für nicht-endliche Moduln stimmt diese Aussage im Allgemeinen nicht. Die eingangs erwähnte Frage lautet: Stimmt die Aussage für Moduln der Form $D(\mathrm{H}^h_{(x_1,\ldots,x_h)R}(R))$? Die Antwort ist (unter gewissen Voraussetzungen) positiv:

1.3.1 Satz

Seien $(R, \mathfrak{m})$ ein noetherscher lokaler Ring, $h \geq 1$ und $I \subseteq R$ ein Ideal mit $\mathrm{H}_I^h(R) \neq 0 \iff l = h$. Weiter seien $1 \leq h' \leq h$ und $r_1, \ldots, r_{h'} \in I$ eine R-reguläre Folge, die auch $D(\mathrm{H}_I^h(R))$-regulär ist. Es seien $r'_1, \ldots, r'_{h'} \in I$ mit $(r_1, \ldots, r_{h'})R = (r'_1, \ldots, r'_{h'})R$. Dann ist auch $r'_1, \ldots, r'_{h'}$ eine $D(\mathrm{H}_I^h(R))$-reguläre Folge.

In Abschnitt 1.4 ist R_0 ein lokaler Unterring von R und wir untersuchen Beziehungen zwischen

$$D_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R))$$

und

$$D_{R_0}(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R)) \quad :$$

Ein Ergebnis ist

1.4.3 Bemerkung (ii), zweite Aussage
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler äquicharakteristischer kompletter Ring mit Koeffizientenkörper k und $\underline{y} = y_1, \ldots, y_i \in R$ eine Folge in R so, dass $R_0 := k[[y_1, \ldots, y_i]]$ $(\subseteq R)$ regulär und i-dimensional ist (dies ist z. B. der Fall, wenn $\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R) \neq 0$ ist). Wenn Vermutung (*) richtig ist, gilt

$$\mathrm{Ass}_R(D_{R_0}(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R))) = \mathrm{Ass}_R(D_R(\mathrm{H}^i_{(y_1,\ldots,y_i)R}(R))) \quad .$$

In Kapitel 2 werden Eigenschaften der Menge

$$\mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)))$$

untersucht ($(R, \mathfrak{m})$ ein noetherscher lokaler Ring, $\underline{x} = x_1, \ldots, x_i$ eine Folge in R). Die verwendeten Methoden sind konstruktiv in dem Sinne, dass zunächst in dem R-Modul

$$E := k[X_1^{-1}, \ldots, X_n^{-1}]$$

(k ein Körper) gewisse Elemente konstruiert werden (Lemmata 2.1 – 2.3); bekanntlich ist E eine R-injektive Hülle von k, falls $R = k[[X_1, \ldots, X_n]]$ eine formale Potenzreihenalgebra über k ist. Ein zentrales Ergebnis in diesem Kapitel (und eine Folgerung aus Lemma 2.1) ist

2.4 Satz
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler äquicharakteristischer Ring, $i \geq 1$ und $x_1, \ldots, x_i$ eine Folge in R. Dann ist

$$\{\mathfrak{p} \in \mathrm{Spec}(R) | x_1, \ldots, x_i \text{ ist Teil eines Paramtersystem von } R/\mathfrak{p}\} \subseteq \mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))) \quad .$$

Satz 2.5 enthält ein ähnliches Ergebnis im gemischt-charakteristischen Fall.

Satz 2.4 ermöglicht es, im Falle $i = 1$ die Menge der assoziierten Primideale vollständig zu berechnen:

2.6 Korollar
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler äquicharakteristischer Ring und $x \in R$. Dann ist

$$\mathrm{Ass}_R(D(\mathrm{H}^1_{xR}(R))) = \mathrm{Spec}(R) \setminus (\mathcal{V}x) \quad .$$

Insbesondere ist die Menge der assoziierten Primideale des Matlis-Duals einen lokalen Kohomologiemoduls im Allgemeinen nicht endlich.

Andererseits zeigen wir in Bemerkung 2.7 (ii), dass die in Satz 2.4 bewiesene Inklusion im Allgemeinen echt ist, dass also nicht alle assoziierten Primideale von $D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))$ von der in Satz 2.4 angegebenen Form sind. Schließlich untersuchen wir (in Bemerkung 2.7 (iii)) die Teilmengen

$$Z_1 := \{\mathfrak{p} \in \mathrm{Spec}(R) | \, \mathrm{H}^i_{(x_1,\ldots,x_i)R}(R/\mathfrak{p}) \neq 0\}$$

und

$$Z_2 := \{\mathfrak{p} \in \mathrm{Spec}(R) | x_1, \ldots, x_i \text{ ist Teil eines Parametersystems von } R/\mathfrak{p}\}$$

von $\mathrm{Spec}(R)$ im Hinblick auf ihre Abgeschlossenheit unter Generalisierung (man beachte, dass gemäß Satz 2.4 (bzw. 2.5) und Bemerkung 1.1.2)

$$Z_2 \subseteq \mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))) \subseteq Z_1$$

gilt). Dabei zeigt sich, dass Z_1 abgeschlossen unter Generalisierung ist, Z_2 hingegen im Allgemeinen nicht, selbst dann nicht, wenn R regulär ist. Immerhin gilt die schwächere Aussage

$$Z_2 \neq \emptyset \Longrightarrow \{0\} \in Z_2 \quad .$$

In Kapitel 3 wird die Untersuchung von $\mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)))$ fortgesetzt, wobei nun keine Voraussetzungen über die (Gleich-)Charakteristik gemacht werden. Dabei ist nachfolgendes Lemma ein entscheidender Ausgangspunkt:

3.1.1 Lemma
Seien R ein Ring, $x, y \in R$ und U ein R-Untermodul von R_x mit $\mathrm{im}(\iota_x) \subseteq U$ ($\iota : R \to R_x$ bezeichnet die kanonische Abbildung). Weiter bezeichne $S := \mathrm{im}(\iota_y) \subseteq R_y$. Dann existiert ein Epimorphismus

$$R_x/U \to R_{xy}/(S_x + U_y)$$

von R-Moduln.

Daraus folgt unter Verwendung von Čech-Kohomologie leicht

3.1.2 Satz
Seien R ein noetherscher Ring, $x_1, \ldots, x_m, y_1, \ldots, y_n \in R$ ($m \in \mathbb{N}^+, n \in \mathbb{N}$) und M ein R-Modul. Dann existiert ein Epimorphism

$$\mathrm{H}^m_{(x_1,\ldots,x_m)R}(R) \to \mathrm{H}^{m+n}_{(x_1,\ldots,x_m,y_1,\ldots,y_n)R}(R)$$

von R-Moduln.

Die Idee ist nun, diesen Epimorphismus zu dualisieren; man erhält einen Monomorphismus und folglich eine Inklusionsbeziehung zwischen den jeweiligen Mengen von assoziierten Primidealen. Wir erhalten:

3.1.3 Satz
Seien $(R,\mathfrak{m})$ ein noetherscher lokaler Ring, $m \in \mathbb{N}^+$, $x_1,\dots,x_m \in R$ und M ein endlich erzeugter R-Modul. Dann gelten:
(i) Für jedes $\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^m_{(x_1,\dots,x_m)R}(M)))$ ist $\dim(M/\mathfrak{p}M) \geq m$.
(ii) $\{\mathfrak{p} \in \mathrm{Supp}_R(M) | x_1,\dots,x_m$ ist Teil eines Parametersystem von $R/\mathfrak{p}\} \subseteq \mathrm{Ass}_R(D(\mathrm{H}^m_{(x_1,\dots,x_m)R}(M)))$.
(iii) Für jedes $x \in R$ gilt $\mathrm{Ass}_R(D(\mathrm{H}^1_{xR}(R))) = \mathrm{Spec}(R) \setminus \mathcal{V}(x)$.
(iv) Ist $x_1,\dots,x_m$ Teil eines Parametersystems von M, so gilt $\mathrm{Assh}(M) \subseteq \mathrm{Ass}_R(D(\mathrm{H}^m_{(x_1,\dots,x_m)R}(M)))$; im Falle $m = \dim(M)$ gilt sogar Gleichheit: $\mathrm{Assh}(M) = \mathrm{Ass}_R(D(\mathrm{H}^m_{(x_1,\dots,x_m)R}(M)))$ (dabei ist $\mathrm{Assh}(M)$ definiert als die Menge der höchstdimensionalen zu M assoziierten Primideale).
(v) Falls R komplett ist, gilt für jeded $\mathfrak{p} \in \mathrm{Supp}_R(M)$ mit $\dim(R/\mathfrak{p}) = m$ die Äquivalenz

$$\mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^m_{(x_1,\dots,x_m)R}(M))) \iff x_1,\dots,x_m \text{ ist ein Parametersystem von } R/\mathfrak{p} \quad .$$

Im Abschnitt 3.2 wird die Menge

$$\mathrm{Ass}_R(D(\mathrm{H}_I^{\dim(R)-1}(R)))$$

untersucht; dabei ist I (zunächst) ein beliebiges Ideal von R, wir setzen also nicht voraus, dass I (bis auf Radikal) von $\dim(R)-1$ Elementen erzeugt wird. Die wichtigsten Ergebnisse sind die beiden folgenden Sätze:

3.2.6 Satz
Seien $(R,\mathfrak{m})$ ein noetherscher lokaler d-dimensionaler Ring und $J \subseteq R$ ein Ideal mit $\dim(R/J) = 1$ und $\mathrm{H}^d_J(R) = 0$. Dann gilt

$$\mathrm{Assh}(D(\mathrm{H}^{d-1}_J(R))) = \mathrm{Assh}(R) \quad .$$

3.2.7 Satz
Seien $(R,\mathfrak{m})$ ein noetherscher lokaler kompletter d-dimensionaler Ring und $J \subseteq R$ ein Ideal mit $\dim(R/J) = 1$ und $\mathrm{H}^d_J(R) = 0$. Dann gilt

$$\mathrm{Ass}_R(D(\mathrm{H}^{d-1}_J(R)) = \{P \in \mathrm{Spec}(R) | \dim(R/P) = d-1, \dim(R/(P+J)) = 0\} \cup \mathrm{Assh}(R) \quad .$$

Die Beweise sind etwas technisch und beruhen, unter anderem, auf

3.2.1 Lemma
Seien $(S,\mathfrak{m})$ ein noetherscher lokaler kompletter Gorenstein-Ring der Dimension $n+1$ und $\mathfrak{P} \subseteq S$ ein Primideal der Höhe n. Dann gilt kanonisch

$$D(\mathrm{H}^n_{\mathfrak{P}}(S)) = \widehat{S_{\mathfrak{P}}}/S \quad .$$

In Kapitel 4 untersuchen wir einen Spezialfall, den wie als "regulären Fall"bezeichnen: k ein Körper, $R = k[[X_1,\dots,X_n]]$ eine Potenzreihenalgebra über k in n Variablen und I das Ideal $(X_1,\dots,X_h)R$ von R $(1 \leq h \leq n)$. Zum Beweis von Vermutung (*) kann man sich auf den regulären Fall zurückziehen:

4.1.2 Satz
Seien $(R,\mathfrak{m})$ ein noetherscher lokaler kompletter Ring mit einem Koeffizientenkörper k, $l \in \mathbb{N}^+$ und $x_1,\dots,x_l$ Teil eines Parametersystems von R. $I := (x_1,\dots,x_l)R$. Seien $x_{l+1},\dots,x_d \in R$ so, dass $x_1,\dots,x_d$ ein

Parametersystem von R ist. R_0 bezeichne den (d-dimensionalen, regulären) Unterring $k[[x_1, \ldots, x_d]]$ von R. Ist $\mathrm{Ass}_{R_0}(D(\mathrm{H}^l_{(x_1,\ldots,x_l)R_0}(R_0)))$ abgeschlossen unter Generalisierung, so auch $\mathrm{Ass}_R(D(\mathrm{H}^l_{(x_1,\ldots,x_l)R}(R)))$.

In Abschnitt 4.2 behandeln wir den regulären Fall. Satz 4.2.1 fasst (im Wesentlichen) zusammen, was die bisher gezeigten Sätze im regulären Fall bedeuten. Ein weiteres Ergebnis ist

4.2.3 Satz

Seien $(R_0, \mathfrak{m}_0)$ ein noetherscher lokaler kompletter äquicharakteristischer Ring, $\dim(R_0) = n-1$, $k \subseteq R_0$ ein Koeffizientenkörper und $h \in \{1, \ldots, n\}$. Weiter seien $x_1, \ldots, x_n \in R$ Elemente mit $\sqrt{(x_1, \ldots, x_n)R} = \sqrt{\mathfrak{m}_0}$. $I_0 := (x_1, \ldots, x_h)R_0$. $R := k[[X_1, \ldots, X_n]]$ sei eine Potenzreihenalgebra über k in n Unbestimmten, $I := (X_1, \ldots, X_h)R$. Der durch $X_i \mapsto x_i$ $(i = 1, \ldots, n)$ festegelegte k-Algebrahomomorphismus $R \to R_0$ induziert einen modul-endlichen Homomorphismus $\iota : R/fR \to R_0$ mit einem geeigneten Primideal f von R. Wir setzen

$$D := D(\mathrm{H}^h_I(R)) \quad .$$

Dann gelten:

(i) D hat ein f enthaltendes assoziiertes Primideal genau dann, $\mathrm{H}^h_{I_0}(R_0) \neq 0$ gilt.

Wenn wir (zusätzlich) annehmen, dass R_0 regulär ist und dass $\mathrm{height}(I_0) < h$ ist, so gelten:

(ii) Es gibt keine zu D assoziiertes Primideal, dass f enthält und die Höhe $n-h$ hat.

(iii) Wenn $\mathrm{H}^h_{I_0}(R_0) \neq 0$ ist (dann ist f in einem zu D assoziierten Primideal enthalten), gilt $\dim(R/\mathfrak{q}) > h$ für jedes f enthaltende maximale Element in $\mathrm{Ass}_R(D)$.

Aussage (iii) hängt eng mit dem Gegenbeispiel zur Vermutung (+) aus [HS1, Kapitel 0] zusammen – vgl. dazu den Abschnitt nach Satz 1.2.3 in dieser Zusammenfassung.

Abschnitt 4.3 behandelt den Fall $h = n-2$ (in den Fällen $h = n-1$ und $h = n$ wurde $\mathrm{Ass}_R(D(\mathrm{H}^h_I(R)))$ vollständig bestimmt – vgl. Satz 4.2.1): Unter anderem zeigen wir:

4.3.1 Korollar

In der Situation von Satz 4.2.3 seien R_0 regulär und $\mathrm{height}(I_0) < n-2 =: h$. Dann gilt

$$fR \in \mathrm{Ass}_R(D) \iff \mathrm{H}^{n-2}_{I_0}(R_0) \neq 0 \quad .$$

Falls dieses Bedingungen zutreffen, ist fR maximal in $\mathrm{Ass}_R(D)$.

Bekanntlich (vgl. [HL, Theorem 2.9) ist $\mathrm{H}^{n-2}_{I_0}(R_0)$ genau dann trivial, wenn $\mathrm{Spec}(R_0/I_0) \setminus \{\mathfrak{m}_0/I_0\}$ formalgeometrisch zusammenhängend ist.

Der folgende Satz ist für sich genommen interessant und wird sich später (im Abschnitt 6.2: Verallgemeinerung eines Beispiels von Hartshorne) als nützlich erweisen:

4.3.4 Satz

Es seien k ein Körper, $R = k[[X_1, \ldots, X_n]]$ $(n \geq 3)$ eine Potenzreihenalgebra über k in n Variablen und I das Ideal $(X_1, \ldots, X_{n-2})R$. Ausserdem sei $p \in R$ ein Primelement mit $p \in I \cap (X_{n-1}, X_n)R$. Dann ist die Menge

$$\{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p} \in \mathrm{Ass}_R(D(\mathrm{H}^{n-2}_I(R))), p \in \mathfrak{p}, \mathrm{height}(\mathfrak{p}) = 2\}$$

unendlich.

Kapitel 5 behandelt die Frage: Was bedeutet es, dass ein gegebenes Ideal arithmetischen Rang eins oder zwei hat? Unter anderem werden Kriterien für diese Bedingungen bewiesen. Zu Beginn jedoch präsentieren wir ein Beispiel, bei dem arithmetischer Rang und kohomologische Dimension nicht übereinstimmen:

5.1 Beispiel
Seien k ein Körper und $R = k[[x,y,z,w]]$ eine Potenzreihenalgebra über k in 4 Variablen. Es sei

$$I := \sqrt{(xw - yz, y^3 - x^2z, z^3 - w^2y)R} \quad .$$

Dann gilt

$$\mathrm{cd}(I/(xw-yz)R) = 1 \neq 2 = \mathrm{ara}(I/(xw-yz)R) \quad .$$

Die Hauptergebnisse in Abschnitt 5.2 sind Kriterien für $\mathrm{ara}(I) \leq 1$ bzw. $\mathrm{ara}(I) \leq 2$:

Definition
Seien $(R,\mathfrak{m})$ ein noetherscher lokaler Ring und X eine Teilmenge von $\mathrm{Spec}(R)$. Wir sagen, dass X Primvermeidung erfüllt, wenn für jedes Ideal J von R die Implikation

$$J \subseteq \bigcup_{\mathfrak{p} \in X} \mathfrak{p} \Longrightarrow \exists \mathfrak{p}_0 \in X : J \subseteq \mathfrak{p}_0$$

gilt.

5.2.5 Satz (i)
Es sei I ein Ideal in einem noetherschen lokalen Ring $(R,\mathfrak{m})$ mit $0 = \mathrm{H}^2_I(R) = \mathrm{H}^3_I(R) = \ldots$. Dann gilt:

$$\mathrm{ara}(I) \leq 1 \iff \mathrm{Ass}_R(D(\mathrm{H}^1_I(R))) \text{ erfüllt Primvermeidung} \quad .$$

5.2.6 Korollar (i)
Sei I ein Ideal in einem noetherschen lokalen Ring $(R,\mathfrak{m})$. Genau dann gilt $\mathrm{ara}(I) \leq 2$, wenn ein $g \in I$ existiert mit $0 = \mathrm{H}^2_I(R/gR) = \mathrm{H}^3_I(R/gR) = \ldots$ und so, dass $\mathrm{Ass}_R(D(\mathrm{H}^1_I(R/gR)))$ Primvermeidung erfüllt.

Zu beiden Kriterien gibt es analoge Aussagen im graduierten Fall (Satz 5.2.5 (ii) und Korollar 5.2.6 (ii)), auch die Beweise sind analog.

Abschnitt 5.3 behandelt subtile Unterschiede zwischen der graduierten und der lokalen Situation.

Kapitel 6 enthält verschiedene Anwendungen der in den vorangehenden Kapitel entwickelten Theorie: Als erste Anwendung verweisen wir auf zwei neue Beweise (Satz 6.1.2 und Satz 6.1.4) des Satzes von Hartshorne-Lichtenbaum; der besagt bekanntlich, dass für einen noetherschen lokalen kompletten Integritätsring $(R,\mathfrak{m})$ und ein Ideal $I \subseteq R$ genau dann $\mathrm{H}^{\dim(R)}_I(R) \neq 0$ gilt, wenn $\sqrt{I} = \mathfrak{m}$ ist. Der Beweis von 6.1.2 verwendet die Normalisierung von R und Matlis-Duale von lokalen Kohomologiemoduln; beim zweiten Beweis (6.1.4) verwenden wir die Tatsache, dass über einem noetherschen lokalen kompletten Gorenstein-Ring $(S,\mathfrak{m})$ mit $\dim(S) = n+1$ für jedes Primideal $\mathfrak{P}$ von S der Höhe n

$$D(\mathrm{H}^n_{\mathfrak{P}}(S)) = \widehat{S_{\mathfrak{P}}}/S$$

gilt (dies ist Lemma 3.2.1); besonders bemerkenswert ist dabei wohl, dass der Beweis von 6.1.4 die Ring-Struktur von $\widehat{S_{\mathfrak{P}}}$ verwendet (nämlich im Beweis von Lemma 6.1.3).

Hartshorne ([Ha1, section 3]) untersuchte (im Wesentlichen) folgendes Beispiel: Seien k ein Körper, $R = k[[X_1, X_2, X_3, X_4]]$ eine Potenzreihenalgebra über k in vier Unbestimmten, $I = (X_1, X_2)R$ und $p = X_1X_4 + X_2X_3 \in R$. Dann ist $\operatorname{Supp}_R(\mathrm{H}^2_I(R/pR)) = \{\mathfrak{m}\}$, aber $\mathrm{H}^2_I(R/pR)$ ist nicht artinsch als R-Modul. In Abschnitt 6.2 der vorliegenden Arbeit zeigen wir zunächst, dass $D(\mathrm{H}^2_I(R/pR))$ unendliche viele assoziierte Primideale hat; somit ist $\mathrm{H}^2_I(R/pR)$ nicht artinsch. Mit anderen Worten: Die Untersuchung der assoziierten Primideale von $D(\mathrm{H}^2_I(R/pR))$ führt zu einem einfachen Beweis der Tatsache, dass $\mathrm{H}^2_I(R/pR)$ nicht artinsch ist. Diese Idee wird nun verallgemeinert zu:

6.2.3 Satz
Seien k ein Körper, $n \geq$, $R = k[[X_1, \ldots, X_n]]$, $I = (X_1, \ldots, X_{n-2})R$ und p ein Primelement in R mit $p \in (X_{n-1}, X_n)R$. Dann ist

$$\mathrm{H}^{n-2}_I(R/pR)$$

nicht artinsch.

Auch Marley und Vassilev ([MV, theorem 2.3]) haben Hartshornes Beispiel verallgemeinert; man kann [MV, theorem 2.3] und Satz 6.2.3 nur in einem Spezialfall vergleichen: Dies machen wir in Bemerkung 6.2.5 und erhalten als Ergebnis, dass (in diesem Spezialfall) Satz 6.2.3 mit schwächeren Voraussetzungen auskommt als [MV, theorem 2.3].

Im Abschnitt 6.3 ist $(R, \mathfrak{m})$ ein noetherscher lokaler Ring. Ist nun $I = (x_1, \ldots, x_i)R \subseteq R$ ein Ideal, das mengentheoretisch vollständiger Durchschnitt ist (im Sinne von $\operatorname{height}(I) = i$), so ergibt sich sofort $\mathrm{H}^i_I(R) \neq 0$, z. B. indem man lokalisiert. Hauptergebnis in 6.3 ist nun eine gewisse notwendige Bedingung für $\mathrm{H}^i_I(R) \neq 0$:

6.3.1 Satz (partiell)
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler kompletter Integritätsring, der einen Körper k enthalte und $x_1, \ldots, x_i \in R$ $(i \geq 1)$ eine Folge in R. Bezeichne R_0 den Unterring $k[[x_1, \ldots, x_i]]$ von R. Dann gilt die Implikation

$$\mathrm{H}^i_I(R) \neq 0 \Longrightarrow R \cap Q(R_0) = R_0$$

(dabei ist $Q(R_0)$ der Quotientenkörper von R_0 und die Durchschnittsbildung ist in $Q(R)$ gemeint).

Über gewissen Ringen (z. B. kompletten Cohen-Macaulay Ringen) gibt es eine Korrespondenz zwischen Ext-Moduln auf der einen und lokalen Kohomologiemoduln auf der anderen Seite; diese wird als lokale Dualität bezeichnet, vgl. dazu etwa [BS, section 11]. Alle in dieser Korrespondenz vorkommenden lokalen Kohomologiemoduln haben $\mathfrak{m}$ als Trägerideal. In Abschnitt 6.4 verallgemeinern wir dieses Prinzip auf beliebige Ideale:

6.4.1 Satz
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler Ring, $I \subseteq R$ ein Ideal, $h \in \mathbb{N}$ mit

$$\mathrm{H}^l_I(R) \neq 0 \iff l = h$$

und sei M ein R-Modul. Dann gilt für jedes $i \in \{0, \ldots, h\}$ kanonisch

$$\mathrm{Ext}^i_R(M, D(\mathrm{H}^h_I(R))) = D(\mathrm{H}^{h-i}_I(M)) \quad .$$

Die nachfolgende Bemerkung 6.4.2 zeigt, dass Satz 6.4.1 wirklich eine verallgemeinerte lokale Dualität ist.

In Abschnitt 7.2 zeigen wir, dass $D(\mathrm{H}^i_I(R))$ eine kanonische D-Modul-Struktur hat; damit ist folgendes gemeint: Seien k ein Körper und $R = k[[X_1, \ldots, X_n]]$ eine Potenzreihenalgebra über k in n Unbestimmten. Sei

$$D(R, k) \subseteq \mathrm{End}_k(R)$$

der (nicht-kommutative) Unterring, der von allen Multiplikationen mit allen Elementen aus R und allen k-linearen Derivationen erzeugt wird. $D := D(R, k)$ wird als Ring der k-linearen Derivationen auf R bezeichnet. (Links-)D-Moduln im Zusammenhamg mit lokaler Kohomologie wurden in [Ly1] studiert; darin wurde auch gezeigt, dass (in sehr allgemeinen Situationen) lokale Kohomologiemoduln eine kanonische (Links-)D-Modul-Struktur tragen. Wir zeigen nun (in 7.2), dass für jedes Ideal $I \subseteq R = k[[X_1, \ldots, X_n]]$ und für jedes $i \in \mathbb{N}$ auch

$$D(\mathrm{H}^i_I(R))$$

eine kanonische (Links-)D-Modul-Struktur hat; weiter zeigen wir, dass $D(\mathrm{H}^i_I(R))$ als D-Modul im Allgemeinen nicht endlich erzeugt ist, insbesondere nicht holonom (siehe [Bj, sections 1,3] für den Begriff der holonomen D-Moduln).

Seien $(R, \mathfrak{m})$ ein noetherscher lokaler Integritätsring und $x_1, \ldots, x_i \in R$ $(i \geq 1)$. In zahlreichen Situationen (vgl. etwa Satz 3.1.3 (ii)) gilt dann

$$\{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)))$$

(sogar immer falls $\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R) \neq 0$, wenn Vermutung (*) zutrifft). Es ist natürlich, nach der $Q(R)$-Vektorraum-Dimension von

$$D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)) \otimes_R Q(R)$$

zu fragen (dies ist eine sogenannte Bass-Zahl von $D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))$). Es zeigt sich, dass diese Dimension im Allgemeinen nicht endlich ist; genauer gilt:

7.3.2 Satz
Seien k ein Körper und $R = k[[X_1, \ldots, X_n]]$ eine Potenzreihenalgebra über k in $n \geq 2$ Unbestimmten, $1 \leq i < n$ und I das Ideal $(X_1, \ldots, X_i)R$ von R. Dann ist

$$\dim_{Q(R)}(D(\mathrm{H}^i_I(R)) \otimes_R Q(R)) = \infty \quad .$$

In Abschnitt 7.4 untersuchen wir Moduln der Form $\mathrm{H}^h_I(D(\mathrm{H}^h_I(R)))$. Das Hauptergebnis ist

7.4.1 Satz und 7.4.2 Satz (Spezialfall)
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler kompletter regulärer Ring der Äquicharakteristik Null, $I \subseteq R$ ein Ideal der Höhe $h \geq 1$ mit $\mathrm{H}^l_I(R) = 0$ $(l > h)$; weiter sei $\underline{x} = x_1, \ldots, x_h$ eine R-reguläre Folge in I. Dann ist

$$\mathrm{H}^h_I(D(\mathrm{H}^h_I(R)))$$

entweder gleich Null oder isomorph zu $\mathrm{E}_R(R/\mathfrak{m})$. Im Falle $I = (x_1, \dots, x_h)R$ trifft letzteres zu, i. e.

$$\mathrm{H}^h_I(D(\mathrm{H}^h_I(R))) = \mathrm{E}_R(R/\mathfrak{m}) \quad .$$

In den Abschnitten 8.1 und 8.2 werden sogenannte "attached" Primideale studiert, und zwar im Hinblick auf lokale Kohomologiemoduln; 8.1 versammelt zahlreiche grundlegende (und teilweise natürlich bekannte) Eigenschaften von "attached" Primidealen, 8.2 enthält unsere Ergebnisse, d. h. Informationen über "attached" Primideale von lokalen Kohomoliemoduln. Schon in 8.1 zeigt sich ein enger Zusammenhang zwischen assoziierten den assoziierten Primidealen vom Matlis-Dual eines R-Moduls M einerseits und den "attached" Primidealen von M andererseits. Hier eine Auswahl unserer Ergebnisse:

8.2.1 Satz
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler Ring und M ein endlich erzeugter n-dimensionaler R-Modul. Dann gilt

$$\mathrm{Att}_R(\mathrm{H}^n_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) |\, \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = n\} \quad .$$

Dies war ursprünglich ein Ergebnis von Dibaei und Yassemi ([DY, Theorem A], vgl. auch [MS, theorem 2.2]); hier wird es mit anderen Methoden bewiesen. Neue Ergebnisse sind

8.2.3 Satz
Es sei $(R, \mathfrak{m})$ ein noetherscher lokaler d-dimensionaler Ring.
(i) Ist J ein Ideal von R mit $\dim(R/J) = 1$ und $\mathrm{H}^d_J(R) = 0$, so gilt

$$\mathrm{Assh}(R) \subseteq \mathrm{Att}_R(\mathrm{H}^{d-1}_J(R)) \quad .$$

Ist R (zusätzlich) komplett, so gilt sogar

$$\mathrm{Att}_R(\mathrm{H}^{d-1}_J(R)) = \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \dim(R/\mathfrak{p}) = d-1, \sqrt{\mathfrak{p}+J} = \mathfrak{m}\} \cup \mathrm{Assh}(R) \quad .$$

(ii) Für jede Folge $x_1, \dots, x_i$ in R gilt

$$\{\mathfrak{p} \in \mathrm{Spec}(R) | x_1, \dots, x_i \text{ ist Teil eines Parametersystems von } R/\mathfrak{p}\} \subseteq \mathrm{Att}_R(\mathrm{H}^i_{(x_1,\dots,x_i)R}(R)) \quad .$$

8.2.4 Korollar
Sei $(R, \mathfrak{m})$ ein noetherscher lokaler Ring. Dann gilt für jedes $x \in R$

$$\mathrm{Att}_R(\mathrm{H}^1_{xR}(R)) = \mathrm{Spec}(R) \setminus \mathcal{V}(x) \quad .$$

Die weitere Untersuchung zeigt, dass ein Ergebnis aus Abschnitt 8.1 (nämlich Satz 8.1.13) als zusätzliche Evidenz für unsere Vermutung (*) aufgefasst werden kann; die Details dazu sind etwas technisch – vgl. Bemerkung 8.2.6 (iii) (ζ).

Es gibt eine Theorie der lokalen Homologie ([T1], [T2]); diese ist in gewisser Weise dual zur lokalen Kohomologietheorie. Seien $(R, \mathfrak{m})$ ein noetherscher lokaler Ring, $\underline{x} = x_1, \dots, x_r$ eine Folge in $\mathfrak{m}$ und X ein artinscher R-Modul. Dann ist der i-te lokale Homologiemodul $\mathrm{H}^{\underline{x}}_i(X)$ von X bezüglich $\underline{x}$ definiert als

$$\varprojlim_{n \in \mathbf{N}} H_i(K_\bullet(x_1^n, \dots, x_r^n; X)) \quad ,$$

wobei $K_\bullet(x_1^n, \ldots, x_r^n; X)$ der Koszul-Komplex von X bezüglich $x_1^n, \ldots, x_r^n$ ist und wobei H_i für die i-te Homologie dieses Komplexes steht. Man beachte, dass diese Homologien bezüglich n in naheliegender Weise ein projektives System bilden. Es ist leicht zu sehen, dass $\mathrm{H}_I^{\underline{x}}$ ein R-linearer kovarianter Funktor von der Kategorie der artinschen R-Moduln in die Kategorie der R-Moduln ist. Den Begriffen der (Krull-) Dimension und der Tiefe (von noetherschen, also endlich erzeugten Moduln) entsprechen hier die Begriffe der noetherschen Dimension $\mathrm{N.dim}(X)$ und der Weite $\mathrm{width}(X)$ eines artinschen R-Moduls X: Für $X = 0$ setzt man $\mathrm{N.dim}(X) = -1$, andernfalls bezeichnet $\mathrm{N.dim}(X)$ die kleinste Zahl $r \in \mathbb{N}$, zu der $x_1, \ldots, x_r \in \mathfrak{m}$ mit

$$\mathrm{length}(0 : X(x_1, \ldots, x_r)R) < \infty$$

existieren. Eine Folge $\underline{x} = x_1, \ldots, x_n \in \mathfrak{m}$ heisst X-koregulär, wenn für jedes $i = 1, \ldots, n$

$$(0 : X(x_1, \ldots, x_{i-1})R) \stackrel{x_i}{\to} (0_X(x_1, \ldots, x_{i-1})R)$$

surjektiv ist. $\mathrm{width}(X)$ ist definiert als die Länge (irgend)einer maximalen X-koregulären Folge. [Oo] und [Ro] sind Referenzen für diese Begriffe. Allgemein gilt

$$\mathrm{width}(X) \leq \mathrm{N.dim}(X) < \infty$$

für jeden artinschen R-Modul X; man nennt X ko-Cohen-Macaulay, wenn $\mathrm{width}(X) = \mathrm{N.dim}(X)$ gilt.

Sei M ein endlich erzeugter Cohen-Macaulay R-Modul. Dann ist $\mathrm{H}_\mathfrak{m}^{\dim(M)}(M)$ ko-Cohen-Macaulay mit $\mathrm{N.dim}(\mathrm{H}_\mathfrak{m}^{\dim(M)}(M)) = \dim(M)$ ([T1, Proposition 2.6]). Ausserdem gilt

$$\mathrm{H}_{\dim(M)}^{x_1,\ldots,x_d}(\mathrm{H}_\mathfrak{m}^{\dim(M)}(M)) = \hat{M}$$

(wobei $x_1, \ldots, x_d$ ein Parametersystem für M sei). Seien nun X ein artinscher R-Modul mit $\mathrm{N.dim}(X) = d$ und $\underline{x} = x_1, \ldots, x_d \in \mathfrak{m}$ so, dass $(0 :_X \underline{x})$ endliche Länge hat. Tang stellt die Frage nach der endlichen Erzeugbarkeit von $\mathrm{H}_d^{\underline{x}}(X)$ ([T1, Remark 3.5]). Wir zeigen zunächst mit einem Gegenbeispiel (8.3.1), dass diese Antwort negativ zu beantworten ist; die anschließende Bemerkung 8.3.2 beantwortet – unter zusätzlichen Vorausstzungen – die Frage dann vollständig:

8.3.2 Bemerkung
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler regulärer d-dimensionaler Ring, X ein artinscher ko-Cohen-Macaulay R-Modul mit $\mathrm{N.dim}(X) = d$ und $\underline{x} = x_1, \ldots, x_d \in \mathfrak{m}$ so, dass $(0_X(x_1, \ldots, x_d)R)$ endliche Länge hat. Dann gilt

$$\mathrm{H}_d^{\underline{x}}(X) \text{ ist endlich erzeut als } R\text{-Modul} \iff R \text{ ist komplett.}$$

In einer allgemeineren Situation gilt

8.3.3 Satz
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler kompletter Ring und X ein artinscher R-Modul mit $\mathrm{N.dim}(X) = d$; seien $x_1, \ldots, x_d \in \mathfrak{m}$ so, dass $(0 :_X (x_1, \ldots, x_d)R)$ endliche Länge hat. Dann ist $\mathrm{H}_d^{\underline{x}}(X)$ als R-Modul endlich erzeugt.

Aus Satz 8.3.3 zusammen mit [T1, Remark 3.5] folgt leicht

8.3.4 Korollar
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler kompletter Ring und X ein ko-Cohen-Macaulay R-Modul, $\mathrm{N.dim}(X) = d$; seien $x_1, \ldots, x_d \in \mathfrak{m}$ so, dass $(0 : X(x_1, \ldots, x_d)R)$ endliche Länge hat. Dann ist $\mathrm{H}_d^{x_1,\ldots,x_d}(X)$ Cohen-Macaulay (insbesondere endlich erzeugt). Im Falle $d = \dim(R)$ ist also $\mathrm{H}_d^{x_1,\ldots,x_d}(X)$ ein maximaler Cohen-Macaulay R-Modul.

Nun ordnen wir jedem endlich erzeugten R-Modul M ordnen wir den (artinschen) R-Modul

$$F_2(M) = \mathrm{H}_{\mathfrak{m}}^{\dim(M)}(M)$$

und jedem artinschen R-Modul Xden (endlich erzeugten, 8.3.3) R-Modul

$$G_2(X) = \mathrm{H}^{x_1,\ldots,x_{\mathrm{N.dim}(X)}}(X))$$

zu. F_2 bzw. G_2 induzieren Abbildungen von der Menge der Isomorphieklassen aller noetherschen in die Menge der Isomorphieklassen aller artinschen Moduln bzw. umgekehrt. Auf der anderen Seite induziert auch der Matlis-Dualitätsfunktor D Abbildungen zwischen diesen beiden Mengen. Eine Untersuchungen der Beziehungen zwischen diesen vier Abbildungen (Anmerkungen nach 8.3.4 und Satz 8.3.5) liefert das Ergebnis

8.3.6 Korollar, Aussage (ii)
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler kompletter Ring und I ein Ideal von R mit $I \subseteq \mathrm{Ann}_R(M)$, $\dim(R/I) = \dim(M)$ und so, dass R/I Gorenstein ist. Dann gilt

$$M \text{ ist Cohen-Macaulay} \Longrightarrow \mathrm{Hom}_R(M, R/I) \text{ ist Cohen-Macaulay.}$$

Unterabschnitt 8.4 ist eine weitere Anwendung des Zusammenspiels der weiter oben erwähnten vier Abbildungen. Wir definieren zunächst den Begriff einer Cohen-Macaulayfizierung:

8.4.1 Definition
Seien $(R, \mathfrak{m})$ ein noetherscher lokaler kompletter Ring und M ein endlich erzeugter R-Modul. Ein Obermodul $\tilde{M}$ von M heisst Cohen-Macaulayfizierung von M, wenn folgende drei Bedingungen gelten:
(i) $\tilde{M}$ ist Cohen-Macaulay.
(ii) $\dim(\tilde{M}) = \dim(M)$.
(iii) $\dim(\tilde{M}/M) \leq \dim M - 2$ (diese Bedingung ist äquivalent zu $\mathrm{H}_{\mathfrak{m}}^{\dim(M)-1}(\tilde{M}/M) = \mathrm{H}_{\mathfrak{m}}^{\dim(M)}(\tilde{M}/M) = 0)$.

8.4.2 Satz
Jede Cohen-Macaulayfizierung von M (falls existent) ist zu $(G_2 \circ F_2)(M)$ isomorph.

In [Go] wird ein anderes Konzept des Begriffes "Cohen-Macaulayfizierung" verwendet. Unser Begriff ist eine Verallgemeinerung dieses Konzeptes (siehe Bemerkung 8.4.3 und Satz 8.4.5 in der vorliegenden Arbeit). Abschließend behandeln 8.4.6 und 8.4.7 (einfache) Beispiele von Cohen-Macaulayfizierungen.

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Habilitationsschrift selbständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textestellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellten Materialien oder erbrachten Dienstleistungen als solche gekennzeichnet.

Leipzig, den 20.05.2006

(Dr. Michael Hellus)