The Dynamics of the Jones R&D Growth Model: Technical Appendix

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This paper gives the proofs omitted in my *Review of Economic Dynamics* paper on "The Dynamics of the Jones R&D Growth Model". Arabic equation numbers refer to this paper.

Steady state

From (2) and (3),

$$\chi l = \chi \nu + n \tag{A.1}$$

$$\chi l = (\chi - \alpha)\nu + \alpha^2 z.$$

Hence,

$$\alpha^2 z = \alpha \nu + n. \tag{A.2}$$

From (4) and (A.1):

$$\left(1 - \frac{1-\alpha}{\alpha}\chi\right) l + \gamma = \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu + (1-\alpha) z$$

$$\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\left(\nu + \frac{n}{\chi}\right) + \gamma = \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu + (1-\alpha) z$$

$$\left(\frac{1}{\chi} - \frac{1-\alpha}{\alpha}\right) n + \gamma = \nu + (1-\alpha) z.$$

From (1),

$$\frac{\sigma-1}{\sigma} n - \frac{\rho}{\sigma} + \gamma = \left(1 - \frac{\alpha^2}{\sigma}\right) z. \tag{A.3}$$

Hence,

$$\left(\frac{\sigma-1}{\sigma} - \frac{1}{\chi} + \frac{1-\alpha}{\alpha}\right) n - \frac{\rho}{\sigma} = -\nu + \left(\frac{1}{\alpha} - \frac{1}{\sigma}\right)\alpha^2 z. \tag{A.4}$$

(A.2) and (A.4) can be solved for ν and z:

$$\left(\frac{\sigma-1}{\sigma} - \frac{1}{\chi} + \frac{1-\alpha}{\alpha}\right) n - \frac{\rho}{\sigma} = -\nu + \left(\frac{1}{\alpha} - \frac{1}{\sigma}\right)(\alpha\nu + n)$$

$$\begin{aligned} \nu &= \frac{1}{\alpha}\left(\frac{\sigma-1}{\chi} n + \rho + \frac{1}{\chi} n\right) \\ &= \frac{1}{\alpha}\left(\Delta + \frac{1}{\chi} n\right) \end{aligned}$$

$$\begin{aligned} z &= \frac{\nu}{\alpha} + \frac{n}{\alpha^2} \\ &= \frac{1}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi} n\right). \end{aligned}$$

These are (7) and (8) in the main text. (6) is obtained from (A.1):

$$\begin{aligned} l &= \nu + \frac{n}{\chi} \\ &= \frac{1}{\alpha}\left(\Delta + \frac{n}{\chi}\right) + \frac{n}{\chi} \\ &= \frac{1}{\alpha}\left(\Delta + \frac{1+\alpha}{\chi}n\right). \end{aligned}$$

Finally, (9) is obtained from (A.3):

$$\begin{aligned} \gamma &= \left(1 - \frac{\alpha^2}{\sigma}\right) z - \frac{\sigma - 1}{\sigma} n + \frac{\rho}{\sigma} \\ &= \left(1 - \frac{\alpha^2}{\sigma}\right) \frac{1}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi}n\right) - \frac{\sigma - 1}{\sigma} n + \frac{\rho}{\sigma} \\ &= \frac{1}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi}n\right) - \frac{1}{\sigma}\left(\Delta + \frac{1+\chi}{\chi}n\right) - \frac{\sigma - 1}{\sigma} n + \frac{\rho}{\sigma} \\ &= \frac{1}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi}n\right) - \frac{1}{\sigma}\left(\frac{\sigma - 1}{\chi}n + \rho + \frac{1+\chi}{\chi}n\right) - \frac{\sigma - 1}{\sigma} n + \frac{\rho}{\sigma} \\ &= \frac{1}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi}n\right) - \frac{1+\chi}{\chi}n \\ &= \frac{1}{\alpha^2}\left[\Delta + (1-\alpha^2)\frac{1+\chi}{\chi}n\right]. \end{aligned}$$

For future reference, notice that l^*, ν^*, z^*, and γ^* satisfy the following relations:

$$l^* > \nu^*, \quad z^* > \frac{\nu^*}{\alpha}, \quad z^* > \gamma^*, \quad \nu^* > \frac{l^*}{1+\alpha}, \quad \gamma^* > (1-\alpha^2)z^*. \tag{A.5}$$

The first three inequalities are obvious. The latter two are easily proved as follows:

$$\nu^* = \frac{1}{\alpha}\left(\Delta + \frac{1}{\chi}n\right) = \frac{1}{\alpha}\frac{(1+\alpha)\Delta + \frac{1+\alpha}{\chi}n}{1+\alpha} > \frac{1}{\alpha}\frac{\Delta + \frac{1+\alpha}{\chi}n}{1+\alpha} = \frac{l^*}{1+\alpha}$$

$$\gamma^* = \frac{1}{\alpha^2}\left[\Delta + (1-\alpha^2)\frac{1+\chi}{\chi}n\right] = \frac{1-\alpha^2}{\alpha^2}\left(\frac{\Delta}{1-\alpha^2} + \frac{1+\chi}{\chi}n\right) > \frac{1-\alpha^2}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi}n\right) = (1-\alpha^2)z^*.$$

Transversality condition

The transversality condition for the households' utility maximization problem is

$$\lim_{t\to\infty} e^{-\rho t}[K(t) + A(t)v(t)]\lambda(t) = 0,$$

where λ is the co-state variable and $K + Av$ is financial wealth. According to the first-order condition for an optimal consumption profile,

$$\lambda = \frac{c^{-\sigma}}{L}.$$

In a steady state, $\dot{A}/A = n/\chi$, $\dot{K}/K = \dot{A}/A + n = n/\chi + n$, $\dot{v}/v = \dot{Y}/Y - \dot{A}/A = n$, $\dot{c}/c = n/\chi$, and $\dot{\lambda}/\lambda = -(\dot{c}/c)/\sigma - n = -\sigma n/\chi - n$. Hence, the transversality condition requires

$$-\rho + \left(\frac{n}{\chi} + n\right) + \left(-\sigma\frac{n}{\chi} - n\right) < 0.$$

Rearranging terms yields (5):

$$\Delta \equiv \frac{\sigma - 1}{\chi}n + \rho > 0.$$

The linearized system (10):

From (2):

$$\frac{\partial \dot{l}}{\partial l} = \underbrace{[-\chi(l-\nu)+n]}_{=0} - \chi l = -\chi l.$$

$$\frac{\partial \dot{l}}{\partial \nu} = \chi l$$

$$\frac{\partial \dot{l}}{\partial z} = 0$$

$$\frac{\partial \dot{l}}{\partial \gamma} = 0.$$

Analogously, from (4):

$$\frac{\partial \dot{\nu}}{\partial l} = -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu$$

$$\frac{\partial \dot{\nu}}{\partial \nu} = \underbrace{\left[-\left(1-\frac{1-\alpha}{\alpha}\chi\right)l + \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu + (1-\alpha)z - \gamma\right]}_{=0} + \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu = \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu$$

$$\frac{\partial \dot{\nu}}{\partial z} = (1-\alpha)\nu$$

$$\frac{\partial \dot{\nu}}{\partial \gamma} = -\nu.$$

From (3):

$$\frac{\partial \dot{z}}{\partial l} = \frac{1-\alpha}{\alpha}\chi z$$

$$\frac{\partial \dot{z}}{\partial \nu} = \frac{1-\alpha}{\alpha}(\alpha - \chi)z = \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)z$$

$$\frac{\partial \dot{z}}{\partial z} = \frac{1-\alpha}{\alpha}\underbrace{[\chi l - (\chi-\alpha)\nu - \alpha^2 z]}_{=0} - \alpha(1-\alpha)z = -\alpha(1-\alpha)z$$

$$\frac{\partial \dot{z}}{\partial \gamma} = 0.$$

From (1)

$$\frac{\partial \dot{\gamma}}{\partial l} = 0$$

$$\frac{\partial \dot{\gamma}}{\partial \nu} = 0$$

$$\frac{\partial \dot{\gamma}}{\partial z} = -\left(1 - \frac{\alpha^2}{\sigma}\right)\gamma$$

$$\frac{\partial \dot{\gamma}}{\partial \gamma} = \underbrace{\left[-\left(1 - \frac{\alpha^2}{\sigma}\right) z + \gamma + \frac{\sigma - 1}{\sigma} n - \frac{\rho}{\sigma}\right]}_{=0} + \gamma = \gamma.$$

The characteristic equation

The characteristic equation is:

$$f(q) = \begin{vmatrix} -\chi l^* - q & \chi l^* & 0 & 0 \\ -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* & \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q & (1-\alpha)\nu^* & -\nu^* \\ \frac{1-\alpha}{\alpha}\chi z^* & \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right) z^* & -\alpha(1-\alpha) z^* - q & 0 \\ 0 & 0 & -\left(1 - \frac{\alpha^2}{\sigma}\right)\gamma^* & \gamma^* - q \end{vmatrix} = 0.$$

Developing the determinant in the characteristic equation with respect to the fourth row gives:

$$f(q) = \left(1 - \frac{\alpha^2}{\sigma}\right)\gamma^* \Delta_1(q) + (\gamma^* - q)\Delta_2(q), \tag{A.6}$$

where

$$\Delta_1(q) = \begin{vmatrix} -\chi l^* - q & \chi l^* & 0 \\ -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* & \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q & -\nu^* \\ \frac{1-\alpha}{\alpha}\chi z^* & \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right) z^* & 0 \end{vmatrix}$$

and

$$\Delta_2(q) = \begin{vmatrix} -\chi l^* - q & \chi l^* & 0 \\ -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* & \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q & (1-\alpha)\nu^* \\ \frac{1-\alpha}{\alpha}\chi z^* & \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right) z^* & -\alpha(1-\alpha) z^* - q \end{vmatrix}.$$

$\Delta_1(q)$ can be written as:

$$\begin{aligned} \Delta_1(q) &= \begin{vmatrix} -\chi l^* - q & \chi l^* & 0 \\ -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* & \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q & -\nu^* \\ \frac{1-\alpha}{\alpha}\chi z^* & \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right) z^* & 0 \end{vmatrix} \\ &= -\frac{1-\alpha}{\alpha}\chi^2 l^* \nu^* z^* - \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* z^* - \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^* q \\ &= -(1-\alpha)\chi l^* \nu^* z^* - \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^* q. \end{aligned} \tag{A.7}$$

As for $\Delta_2(q)$:

$$\begin{aligned}
\Delta_2(q) &= \begin{vmatrix} -\chi l^* - q & \chi l^* & 0 \\ -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* & \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q & (1-\alpha)\nu^* \\ \frac{1-\alpha}{\alpha}\chi z^* & \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right) z^* & -\alpha(1-\alpha)z^* - q \end{vmatrix} \\
&= (-\chi l^* - q)\left[\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q\right][-\alpha(1-\alpha)z^* - q] \\
&\quad + \frac{(1-\alpha)^2}{\alpha}\chi^2 l^* \nu^* z^* + \left(1 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* [-\alpha(1-\alpha)z^* - q] \\
&\quad -(1-\alpha)\left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^* (-\chi l^* - q).
\end{aligned}$$

The first term in this sum can be rewritten as:

$$\begin{aligned}
& (-\chi l^* - q)\left[\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q\right][-\alpha(1-\alpha)z^* - q] \\
= & \left[\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q\right]\{\alpha(1-\alpha)\chi l^* z^* + [\chi l^* + \alpha(1-\alpha)z^*]q + q^2\} \\
= & \alpha(1-\alpha)\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* z^* \\
& + \left[\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* - \alpha(1-\alpha)\chi l^* z^* + \alpha(1-\alpha)\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] q \\
& + \left[-\chi l^* + \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - \alpha(1-\alpha)z^*\right] q^2 \\
& - q^3.
\end{aligned}$$

The remaining terms can be rewritten as:

$$\begin{aligned}
& \frac{(1-\alpha)^2}{\alpha}\chi^2 l^* \nu^* z^* + \left(1 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* [-\alpha(1-\alpha)z^* - q] \\
& -(1-\alpha)\left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^* (-\chi l^* - q) \\
= & (1-\alpha)\chi l^* \nu^* z^* \left[\frac{1-\alpha}{\alpha}\chi - \alpha\left(1 - \frac{1-\alpha}{\alpha}\chi\right) + \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\right] \\
& + \left[-\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* + (1-\alpha)\left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] q \\
= & (1-\alpha)\left[-\alpha\left(1 - \frac{1-\alpha}{\alpha}\chi\right) + 1 - \alpha\right]\chi l^* \nu^* z^* \\
& + \left[-\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* + (1-\alpha)\left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] q.
\end{aligned}$$

Taken together, it follows that

$$\begin{aligned}
\Delta_2(q) &= (1-\alpha)\chi l^* \nu^* z^* \\
&\quad + \left[\chi l^* \nu^* - \alpha(1-\alpha)\chi l^* z^* + (1-\alpha^2)\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] q
\end{aligned}$$

$$
\begin{aligned}
&+\left[-\chi l^* + \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - \alpha(1-\alpha)z^*\right]q^2 \\
&-q^3.
\end{aligned}
\tag{A.8}
$$

Equations (11)-(14):

From (A.6)-(A.8), one can calculate the expressions for δ_i $(i = 0, 1, 2, 3)$ in (11)-(14):

$$
\begin{aligned}
\delta_0 &= -(1-\alpha)\left(1 - \frac{\alpha^2}{\sigma}\right)\chi l^*\nu^* z^*\gamma^* + (1-\alpha)\chi l^*\nu^* z^*\gamma^* \\
&= (1-\alpha)\frac{\alpha^2}{\sigma}\chi l^*\nu^* z^*\gamma^*
\end{aligned}
$$

$$
\begin{aligned}
\delta_1 &= -\left(1 - \frac{\alpha^2}{\sigma}\right)\left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\gamma^* \\
&\quad +\gamma^*\left[\chi l^*\nu^* - \alpha(1-\alpha)\chi l^* z^* + (1-\alpha^2)\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] - (1-\alpha)\chi l^*\nu^* z^* \\
&= \left[-\left(1 - \frac{\alpha^2}{\sigma}\right)\left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right) + (1-\alpha^2)\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\right]\nu^* z^*\gamma^* \\
&\quad -(1-\alpha)\chi l^*\nu^* z^* + \chi l^*\nu^*\gamma^* - \alpha(1-\alpha)\chi l^* z^*\gamma^* \\
&= \alpha(1-\alpha)\left(1 + \frac{\alpha}{\sigma}\right)\nu^* z^*\gamma^* \\
&\quad -\chi\left[(1-\alpha)l^*\nu^* z^* - l^*\nu^*\gamma^* + \alpha(1-\alpha)l^* z^*\gamma^* - \alpha(1-\alpha)\left(1 - \frac{1}{\sigma}\right)\nu^* z^*\gamma^*\right]
\end{aligned}
$$

$$
\begin{aligned}
\delta_2 &= \gamma^*\left[-\chi l^* + \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - \alpha(1-\alpha)z^*\right] \\
&\quad -\left[\chi l^*\nu^* - \alpha(1-\alpha)\chi l^* z^* + (1-\alpha^2)\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] \\
&= 2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha)z^*\gamma^* \\
&\quad -\chi\left[l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*\right]
\end{aligned}
$$

$$
\begin{aligned}
\delta_3 &= -\gamma^* - \left[-\chi l^* + \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - \alpha(1-\alpha)z^*\right] \\
&= \chi l^* - \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* + \alpha(1-\alpha)z^* - \gamma^* \\
&= -2\nu^* + \alpha(1-\alpha)z^* - \gamma^* + \chi\left(l^* + \frac{1-\alpha}{\alpha}\nu^*\right).
\end{aligned}
$$

The term in square brackets in (13) is positive.

From (A.5) and $\mathbf{x}^* > \mathbf{0}$,

$$
l^*\nu^* - \underbrace{\alpha(1-\alpha)}_{=(1-\alpha^2)-(1-\alpha)} l^* z^* + l^* \underbrace{\gamma^*}_{>(1-\alpha^2)z^*} -(1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^* \underbrace{\gamma^*}_{>(1-\alpha^2)z^*}
$$

$$
\begin{aligned}
&> \ l^*\nu^* - (1-\alpha^2)l^*z^* + (1-\alpha)l^*z^* + (1-\alpha^2)l^*z^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^* + (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^* \\
&= \ \underbrace{l^*\nu^*}_{>0} + \underbrace{(1-\alpha)l^*z^*}_{>0} \qquad \text{(A.9)} \\
&> \ 0.
\end{aligned}
$$

Equation (17):

The expression on the right-hand side of (15) is less than the expression on the right-hand side of (16) if

$$
\begin{aligned}
&\frac{\alpha(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\nu^*z^*\gamma^*}{(1-\alpha)l^*\nu^*z^* - l^*\nu^*\gamma^* + \alpha(1-\alpha)l^*z^*\gamma^* - \alpha(1-\alpha)\left(1-\frac{1}{\sigma}\right)\nu^*z^*\gamma^*} \\
< \ &\frac{2\nu^*\gamma^* - (1-\alpha^2)\nu^*z^* - \alpha(1-\alpha)z^*\gamma^*}{l^*\nu^* - \alpha(1-\alpha)l^*z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*}.
\end{aligned}
$$

Since the denominators on both sides of the inequality are positive, this can be rewritten as:

$$
\begin{aligned}
0 \ < \ & [2\nu^*\gamma^* - (1-\alpha^2)\nu^*z^* - \alpha(1-\alpha)z^*\gamma^*] \\
& \times \left[(1-\alpha)l^*\nu^*z^* - l^*\nu^*\gamma^* + \alpha(1-\alpha)l^*z^*\gamma^* - \alpha(1-\alpha)\left(1-\frac{1}{\sigma}\right)\nu^*z^*\gamma^*\right] \\
& -\alpha(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\nu^*z^*\gamma^* \\
& \times \left[l^*\nu^* - \alpha(1-\alpha)l^*z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*\right]
\end{aligned}
$$

or, after division by $(1-\alpha)\nu^*z^*\gamma^*$, as

$$
\begin{aligned}
0 \ < \ & [2\nu^*\gamma^* - (1-\alpha^2)\nu^*z^* - \alpha(1-\alpha)z^*\gamma^*] \\
& \times \left[\frac{l^*}{\gamma^*} - \frac{1}{1-\alpha}\frac{l^*}{z^*} + \alpha\frac{l^*}{\nu^*} - \alpha\left(1-\frac{1}{\sigma}\right)\right] \\
& -\alpha\left(1+\frac{\alpha}{\sigma}\right) \\
& \times \left[l^*\nu^* - \alpha(1-\alpha)l^*z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*\right].
\end{aligned}
$$

Multiplying out yields:

$$
\begin{aligned}
0 \ < \ & 2l^*\nu^* - \frac{2}{1-\alpha}\frac{l^*\nu^*\gamma^*}{z^*} + 2\alpha l^*\gamma^* - 2\alpha\left(1-\frac{1}{\sigma}\right)\nu^*\gamma^* \\
& -(1-\alpha^2)\frac{l^*\nu^*z^*}{\gamma^*} + (1+\alpha)l^*\nu^* - \alpha(1-\alpha^2)l^*z^* + \alpha(1-\alpha^2)\left(1-\frac{1}{\sigma}\right)\nu^*z^* \\
& -\alpha(1-\alpha)l^*z^* + \alpha l^*\gamma^* - \alpha^2(1-\alpha)\frac{l^*z^*\gamma^*}{\nu^*} + \alpha^2(1-\alpha)\left(1-\frac{1}{\sigma}\right)z^*\gamma^* \\
& -\alpha\left(1+\frac{\alpha}{\sigma}\right)l^*\nu^* + \alpha^2(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)l^*z^* - \alpha\left(1+\frac{\alpha}{\sigma}\right)l^*\gamma^* \\
& +(1-\alpha^2)(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\nu^*z^* - (1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\nu^*\gamma^*.
\end{aligned}
$$

Collecting terms, we have:

$$\begin{aligned}
0 &< \left[2+(1+\alpha)-\alpha\left(1+\frac{\alpha}{\sigma}\right)\right] l^*\nu^* \\
&+\left[-\alpha(1-\alpha^2)-\alpha(1-\alpha)+\alpha^2(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\right] l^*z^* \\
&+\left[2\alpha+\alpha-\alpha\left(1+\frac{\alpha}{\sigma}\right)\right] l^*\gamma^* \\
&+\left[\alpha(1-\alpha^2)\left(1-\frac{1}{\sigma}\right)+(1-\alpha^2)(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\right] \nu^*z^* \\
&+\left[-2\alpha\left(1-\frac{1}{\sigma}\right)-(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\right] \nu^*\gamma^* \\
&+\alpha^2(1-\alpha)\left(1-\frac{1}{\sigma}\right) z^*\gamma^* \\
&-(1-\alpha^2)\frac{l^*\nu^*z^*}{\gamma^*}-\frac{2}{1-\alpha}\frac{l^*\nu^*\gamma^*}{z^*}-\alpha^2(1-\alpha)\frac{l^*z^*\gamma^*}{\nu^*}.
\end{aligned}$$

Simplifying terms, we obtain:

$$\begin{aligned}
0 &< \left(3-\frac{\alpha^2}{\sigma}\right) l^*\nu^* \\
&+\alpha(1-\alpha)\left(-2+\frac{\alpha^2}{\sigma}\right) l^*z^* \\
&+\left(2\alpha-\frac{\alpha^2}{\sigma}\right) l^*\gamma^* \\
&+(1-\alpha^2)\left(1-\frac{\alpha^2}{\sigma}\right) \nu^*z^* \\
&+(1+\alpha)\left(-1+\frac{1}{\alpha}\frac{\alpha^2}{\sigma}\right) \nu^*\gamma^* \\
&+(1-\alpha)\left(\alpha^2-\frac{\alpha^2}{\sigma}\right) z^*\gamma^* \\
&-(1-\alpha^2)\frac{l^*\nu^*z^*}{\gamma^*}-\frac{2}{1-\alpha}\frac{l^*\nu^*\gamma^*}{z^*}-\alpha^2(1-\alpha)\frac{l^*z^*\gamma^*}{\nu^*}.
\end{aligned}$$

Collecting terms yields (17):

$$\begin{aligned}
0 &< -\frac{\alpha^2}{\sigma}\left[l^*\nu^*-\alpha(1-\alpha)l^*z^*+l^*\gamma^*+(1-\alpha^2)\nu^*z^*-\frac{1+\alpha}{\alpha}\nu^*\gamma^*+(1-\alpha)z^*\gamma^*\right] \\
&-\Big[-3l^*\nu^*+2\alpha(1-\alpha)l^*z^*-2\alpha l^*\gamma^*-(1-\alpha^2)\nu^*z^*+(1+\alpha)\nu^*\gamma^* \\
&-\alpha^2(1-\alpha)z^*\gamma^*+(1-\alpha^2)\frac{l^*\nu^*z^*}{\gamma^*}+\frac{2}{1-\alpha}\frac{l^*\nu^*\gamma^*}{z^*}+\alpha^2(1-\alpha)\frac{l^*z^*\gamma^*}{\nu^*}\Big].
\end{aligned}$$

The first term in square brackets in (17) is positive:

$$l^*\nu^*-\alpha(1-\alpha)l^*z^*+l^*\gamma^*+(1-\alpha^2)\nu^*z^*-\frac{1+\alpha}{\alpha}\nu^*\gamma^*+(1-\alpha)z^*\gamma^*$$

$$
\begin{aligned}
= \quad & \frac{1}{\alpha^2}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left(\Delta+\frac{1}{\chi}n\right) \\
& -\frac{1-\alpha}{\alpha^2}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left(\Delta+\frac{1+\chi}{\chi}n\right) \\
& +\frac{1}{\alpha^3}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left[\Delta+(1-\alpha^2)\frac{1+\chi}{\chi}n\right] \\
& +\frac{1-\alpha^2}{\alpha^3}\left(\Delta+\frac{1}{\chi}n\right)\left(\Delta+\frac{1+\chi}{\chi}n\right) \\
& -\frac{1+\alpha}{\alpha^4}\left(\Delta+\frac{1}{\chi}n\right)\left[\Delta+(1-\alpha^2)\frac{1+\chi}{\chi}n\right] \\
& +\frac{1-\alpha}{\alpha^4}\left(\Delta+\frac{1+\chi}{\chi}n\right)\left[\Delta+(1-\alpha^2)\frac{1+\chi}{\chi}n\right] \\
= \quad & \Delta^2\frac{1}{\alpha^2}\left[1-(1-\alpha)+\frac{1}{\alpha}+\frac{1-\alpha^2}{\alpha}-\frac{1+\alpha}{\alpha^2}+\frac{1-\alpha}{\alpha^2}\right] \\
& +\Delta\frac{n}{\alpha^2\chi}\Bigg\{1+\alpha+1 \\
& -(1-\alpha)(1+\alpha+1+\chi) \\
& +\frac{1}{\alpha}\left[1+\alpha+(1-\alpha^2)(1+\chi)\right] \\
& +\frac{1-\alpha^2}{\alpha}(1+1+\chi) \\
& -\frac{1+\alpha}{\alpha^2}\left[1+(1-\alpha^2)(1+\chi)\right] \\
& +\frac{1-\alpha}{\alpha^2}[1+\chi+(1-\alpha^2)(1+\chi)]\Bigg\} \\
& +\left(\frac{n}{\alpha\chi}\right)^2\Bigg[1+\alpha \\
& -(1-\alpha)(1+\alpha)(1+\chi) \\
& +\frac{1}{\alpha}(1+\alpha)(1-\alpha^2)(1+\chi) \\
& +\frac{1-\alpha^2}{\alpha}(1+\chi) \\
& -\frac{1+\alpha}{\alpha^2}(1-\alpha^2)(1+\chi) \\
& +\frac{1-\alpha}{\alpha^2}(1-\alpha^2)(1+\chi)+\frac{1-\alpha}{\alpha^2}(1-\alpha^2)\chi(1+\chi)\Bigg] \\
= \quad & \Delta^2\frac{1}{\alpha^2}\left(1-1+\alpha+\frac{1}{\alpha}+\frac{1}{\alpha}-\alpha-\frac{1}{\alpha^2}-\frac{1}{\alpha}+\frac{1}{\alpha^2}-\frac{1}{\alpha}\right) \\
& +\Delta\frac{n}{\alpha^2\chi}\Bigg\{2+\alpha \\
& -2+\alpha+\alpha^2-(1-\alpha)\chi \\
& +\frac{2}{\alpha}+1-\alpha+(1-\alpha)\left(\frac{1}{\alpha}+1\right)\chi
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{\alpha}-2\alpha+(1-\alpha)\left(\frac{1}{\alpha}+1\right)\chi\\
&-\frac{2}{\alpha^2}-\frac{2}{\alpha}+1+\alpha+(1-\alpha)\left(-\frac{1}{\alpha^2}-\frac{2}{\alpha}-1\right)\chi\\
&+\frac{2}{\alpha^2}-\frac{2}{\alpha}-1+\alpha+(1-\alpha)\left(\frac{2}{\alpha^2}-1\right)\chi\Bigg\}\\
&+\left(\frac{n}{\alpha\chi}\right)^2\Bigg[1+\alpha\\
&+(1-\alpha^2)(1+\chi)\left(-1+\frac{1}{\alpha}+1+\frac{1}{\alpha}-\frac{1}{\alpha^2}-\frac{1}{\alpha}+\frac{1}{\alpha^2}-\frac{1}{\alpha}\right)\\
&+\frac{1-\alpha}{\alpha^2}(1-\alpha^2)\chi(1+\chi)\Bigg]\\
=\;&\Delta\frac{n}{\alpha^2\chi}\left[1+\alpha+\alpha^2+(1-\alpha)\frac{1-\alpha^2}{\alpha^2}\chi\right]\\
&+\left(\frac{n}{\alpha\chi}\right)^2(1+\alpha)\left[1+\left(\frac{1-\alpha}{\alpha}\right)^2\chi(1+\chi)\right]\\
>\;&0.
\end{aligned}
$$

Second term in square brackets in (17):

$$
\begin{aligned}
&-3l^*\nu^*+2\alpha(1-\alpha)l^*z^*-2\alpha l^*\gamma^*-(1-\alpha^2)\nu^*z^*+(1+\alpha)\nu^*\gamma^*\\
&-\alpha^2(1-\alpha)z^*\gamma^*+\underbrace{(1-\alpha^2)\frac{l^*\nu^*z^*}{\gamma^*}}_{>(1-\alpha^2)l^*\nu^*}+\underbrace{\frac{2}{1-\alpha}\frac{l^*\nu^*\gamma^*}{z^*}}_{>2(1+\alpha)l^*\nu^*}+\underbrace{\alpha^2(1-\alpha)\frac{l^*z^*\gamma^*}{\nu^*}}_{>\alpha^2(1-\alpha)z^*\gamma^*}\\
>\;&\alpha(2-\alpha)l^*\nu^*+2\alpha(1-\alpha)l^*z^*-2\alpha l^*\gamma^*-(1-\alpha^2)\nu^*z^*+(1+\alpha)\nu^*\gamma^*\\
=\;&\frac{2-\alpha}{\alpha}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left(\Delta+\frac{1}{\chi}n\right)\\
&+\frac{2(1-\alpha)}{\alpha^2}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left(\Delta+\frac{1+\chi}{\chi}n\right)\\
&-\frac{2}{\alpha^2}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left[\Delta+(1-\alpha^2)\frac{1+\chi}{\chi}n\right]\\
&-\frac{1-\alpha^2}{\alpha^3}\left(\Delta+\frac{1}{\chi}n\right)\left(\Delta+\frac{1+\chi}{\chi}n\right)\\
&+\frac{1+\alpha}{\alpha^3}\left(\Delta+\frac{1}{\chi}n\right)\left[\Delta+(1-\alpha^2)\frac{1+\chi}{\chi}n\right]\\
=\;&\Delta^2\frac{1}{\alpha^2}\left[\alpha(2-\alpha)+2(1-\alpha)-2-\frac{1-\alpha^2}{\alpha}+\frac{1+\alpha}{\alpha}\right]\\
&+\Delta\frac{n}{\alpha^2\chi}\Bigg\{\alpha(2-\alpha)(1+\alpha+1)\\
&+2(1-\alpha)(1+\alpha+1+\chi)
\end{aligned}
$$

$$
\begin{aligned}
& -2[1+\alpha+(1-\alpha^2)(1+\chi)] \\
& -\frac{1-\alpha^2}{\alpha}(1+1+\chi) \\
& +\frac{1+\alpha}{\alpha}\left[1+(1-\alpha^2)(1+\chi)\right]\Bigg\} \\
+ \quad & \left(\frac{n}{\alpha\chi}\right)^2\Bigg[\alpha(2-\alpha)(1+\alpha) \\
& +2(1-\alpha)(1+\alpha)(1+\chi) \\
& -2(1+\alpha)(1-\alpha^2)(1+\chi) \\
& -\frac{1-\alpha^2}{\alpha}(1+\chi) \\
& +\frac{1+\alpha}{\alpha}(1-\alpha^2)(1+\chi)\Bigg] \\
= \quad & \Delta^2\frac{1}{\alpha^2}\left(2\alpha-\alpha^2+2-2\alpha-2-\frac{1}{\alpha}+\alpha+\frac{1}{\alpha}+1\right) \\
& +\Delta\frac{n}{\alpha^2\chi}\Bigg[4\alpha-\alpha^3 \\
& +2-2\alpha^2+2(1-\alpha)(1+\chi) \\
& -2-2\alpha+(-2-2\alpha)(1-\alpha)(1+\chi) \\
& -\frac{1}{\alpha}+\alpha+\left(-\frac{1}{\alpha}-1\right)(1-\alpha)(1+\chi) \\
& +\frac{1}{\alpha}+1+\left(\frac{1}{\alpha}+2+\alpha\right)(1-\alpha)(1+\chi)\Bigg] \\
& +\left(\frac{n}{\alpha\chi}\right)^2\Bigg[2\alpha+\alpha^2-\alpha^3 \\
& +\left(2-2-2\alpha-\frac{1}{\alpha}+\frac{1}{\alpha}+1\right)(1-\alpha^2)(1+\chi)\Bigg] \\
= \quad & \Delta^2\frac{1+\alpha-\alpha^2}{\alpha^2} \\
& +\Delta\frac{n}{\alpha^2\chi}\left[1+3\alpha-2\alpha^2-\alpha^3+(1-\alpha)^2(1+\chi)\right] \\
& +\left(\frac{n}{\alpha\chi}\right)^2\left[2\alpha+\alpha^2-\alpha^3+(1-\alpha^2)(1-2\alpha)(1+\chi)\right] \\
= \quad & \Delta^2\frac{1+\alpha-\alpha^2}{\alpha^2} \\
& +\Delta\frac{n}{\alpha^2\chi}\left[2+\alpha-\alpha^2-\alpha^3+\chi(1-\alpha)^2\right] \\
& +\left(\frac{n}{\alpha\chi}\right)^2\left[1+\alpha^3+\chi(1-\alpha^2)(1-2\alpha)\right].
\end{aligned}
$$

The fraction in (18) is greater than 4:

Denote the denominator of the fraction on the right-hand side of (18) as $g(\alpha) \equiv (1-\alpha^2)(2\alpha-1) = -2\alpha^3+\alpha^2+2\alpha-1$. The first and second derivatives are

$$g'(\alpha) = -6\left(\alpha^2 - \frac{1}{3}\alpha - \frac{1}{3}\right)$$

$$g''(\alpha) = -6\left(2\alpha - \frac{1}{3}\right).$$

$g(\alpha)$ assumes a local maximum at $\alpha = (1+\sqrt{13})/6$ with

$$g\left(\frac{1+\sqrt{13}}{6}\right) = \frac{13\sqrt{13}-35}{54}.$$

Since $g(0) = -1 < 0$, this is a global maximum, given $\alpha > 0$. So the denominator is less than $(13\sqrt{13}-35)/54$. For $\alpha > 1/2$, the numerator is greater than $9/8$, and, consequently, the fraction is greater than $243/(52\sqrt{13}-140) = 5.117 > 4$. (Actually, the fraction is greater than 6.464.)

$\chi > 4$ is inconsistent with (16):

Suppose $\chi > 4$ and (16) holds true. Then:

$$4 < \frac{2\nu^*\gamma^* - (1-\alpha^2)\nu^*z^* - \alpha(1-\alpha)z^*\gamma^*}{l^*\nu^* - \alpha(1-\alpha)l^*z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*}$$

Using the fact that $\mathbf{x}^* > \mathbf{0}$, that the denominator is greater than $l^*\nu^* + (1-\alpha)l^*z^*$ (cf. (A.9)), that $2/\alpha < 4$ for $\alpha > 1/2$, and that $2(1+\alpha) < 4$, it follows that

$$\begin{aligned} 4 &< \frac{2\nu^*\gamma^*}{l^*\nu^* - \alpha(1-\alpha)l^*z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*} \\ &< \frac{2\nu^*\gamma^*}{l^*\nu^* + (1-\alpha)l^*z^*} \end{aligned}$$

$$\begin{aligned} 4l^*\nu^* + 4(1-\alpha)l^*z^* - 2\nu^*\gamma^* &< 0 \\ \frac{2}{\alpha}l^*\nu^* + 2(1+\alpha)(1-\alpha)l^*z^* - 2\nu^*\gamma^* &< 0 \\ \frac{2}{\alpha}l^*\nu^* + 2(1-\alpha^2)l^*z^* - 2\nu^*\gamma^* &< 0 \\ \frac{1}{\alpha}l^*\nu^* + (1-\alpha^2)l^*z^* - \nu^*\gamma^* &< 0 \end{aligned}$$

Using (6)-(9), it follows that

$$\begin{aligned} 0 &> \frac{1}{\alpha^3}\left(\Delta + \frac{1+\alpha}{\chi}n\right)\left(\Delta + \frac{1}{\chi}n\right) \\ &+ \frac{1-\alpha^2}{\alpha^3}\left(\Delta + \frac{1+\alpha}{\chi}n\right)\left(\Delta + \frac{1+\chi}{\chi}n\right) \\ &- \frac{1}{\alpha^3}\left(\Delta + \frac{1}{\chi}n\right)\left[\Delta + (1-\alpha^2)\frac{1+\chi}{\chi}n\right] \end{aligned}$$

$$
\begin{aligned}
&= \Delta^2 \frac{1}{\alpha^3}\left[1 + (1-\alpha^2) - 1\right] \\
&\quad +\Delta \frac{n}{\alpha^3\chi}\left\{1 + \alpha + 1 + (1-\alpha^2)(1+\alpha+1+\chi) - \left[1 + (1-\alpha^2)(1+\chi)\right]\right\} \\
&\quad +\left(\frac{n}{\chi}\right)^2 \frac{1}{\alpha^3}\left[1 + \alpha + (1-\alpha^2)(1+\alpha)(1+\chi) - (1-\alpha^2)(1+\chi)\right] \\
&= \Delta^2 \frac{1}{\alpha^3}(1-\alpha^2) \\
&\quad +\Delta \frac{n}{\alpha^3\chi}(1+\alpha)(2-\alpha^2) \\
&\quad +\left(\frac{n}{\chi}\right)^2 \frac{1}{\alpha^3}(1+\alpha)\left[1 + \alpha(1-\alpha)(1+\chi)\right] \\
&> 0,
\end{aligned}
$$

a contradiction.

Lemma 3:

The expression on the right-hand side of (19) is less than the expression on the right-hand side of (16) if

$$
\frac{2\nu^* - \alpha(1-\alpha)z^* + \gamma^*}{l^* + \frac{1-\alpha}{\alpha}\nu^*} < \frac{2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha)z^*\gamma^*}{l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*}.
$$

It has been shown above that the denominator on the right-hand side (i.e., the term in square brackets in (13)) is positive. So this condition can be manipulated as follows:

$$
\begin{aligned}
0 &< \left(l^* + \frac{1-\alpha}{\alpha}\nu^*\right)\left[2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha)z^*\gamma^*\right] \\
&\quad -\left[2\nu^* - \alpha(1-\alpha)z^* + \gamma^*\right]\left[l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*\right] \\
&= 2l^*\nu^*\gamma^* - (1-\alpha^2)l^*\nu^* z^* - \alpha(1-\alpha)l^* z^*\gamma^* \\
&\quad +2\frac{1-\alpha}{\alpha}(\nu^*)^2\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2 z^* - (1-\alpha)^2\nu^* z^*\gamma^* \\
&\quad -2l^*(\nu^*)^2 + 2\alpha(1-\alpha)l^*\nu^* z^* - 2l^*\nu^*\gamma^* + 2(1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2 z^* - 2\frac{1-\alpha}{\alpha}(\nu^*)^2\gamma^* \\
&\quad +\alpha(1-\alpha)l^*\nu^* z^* - \alpha^2(1-\alpha)^2 l^*(z^*)^2 + \alpha(1-\alpha)l^* z^*\gamma^* - (1-\alpha^2)(1-\alpha)^2\nu^*(z^*)^2 + (1-\alpha)^2\nu^* z^*\gamma^* \\
&\quad -l^*\nu^*\gamma^* + \alpha(1-\alpha)l^* z^*\gamma^* - l^*(\gamma^*)^2 + (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^*\gamma^* - \frac{1-\alpha}{\alpha}\nu^*(\gamma^*)^2 \\
&= -2l^*(\nu^*)^2 \\
&\quad +\left[-(1-\alpha^2) + 2\alpha(1-\alpha) + \alpha(1-\alpha)\right]l^*\nu^* z^* \\
&\quad +(2-2-1)l^*\nu^*\gamma^* \\
&\quad +\left[-\alpha^2(1-\alpha)^2\right]l^*(z^*)^2 \\
&\quad +\left[-\alpha(1-\alpha) + \alpha(1-\alpha) + \alpha(1-\alpha)\right]l^* z^*\gamma^* \\
&\quad +(-1)l^*(\gamma^*)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-(1-\alpha^2)\frac{1-\alpha}{\alpha}+2(1-\alpha^2)\frac{1-\alpha}{\alpha}\right](\nu^*)^2 z^* \\
&+\left(2\frac{1-\alpha}{\alpha}-2\frac{1-\alpha}{\alpha}\right)(\nu^*)^2\gamma^* \\
&+[-(1-\alpha^2)(1-\alpha)^2]\nu^*(z^*)^2 \\
&+\left[-(1-\alpha)^2+(1-\alpha)^2+(1-\alpha^2)\frac{1-\alpha}{\alpha}\right]\nu^* z^*\gamma^* \\
&+\left(-\frac{1-\alpha}{\alpha}\right)\nu^*(\gamma^*)^2 \\
= \;& -\Big[2l^*(\nu^*)^2+(1-\alpha)(1-2\alpha)l^*\nu^* z^*+l^*\nu^*\gamma^*+\alpha^2(1-\alpha)^2 l^*(z^*)^2-\alpha(1-\alpha)l^* z^*\gamma^*+l^*(\gamma^*)^2 \\
&-(1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2 z^*+(1-\alpha^2)(1-\alpha)^2\nu^*(z^*)^2-(1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^*\gamma^*+\frac{1-\alpha}{\alpha}\nu^*(\gamma^*)^2\Big].
\end{aligned}
$$

We have to show that the term in square brackets is positive. To do so, we use five preliminary results (obtained using (A.5)):

$$(1-\alpha)(1-2\alpha)l^*\nu^* z^* = (1-\alpha)^2 l^*\nu^* z^* - \alpha(1-\alpha)l^*\nu^* z^* > -\alpha(1-\alpha)l^*\nu^* z^*$$

$$\underbrace{l^*}_{>\nu^*}\nu^* \underbrace{\gamma^*}_{>(1-\alpha^2)z^*} > (1-\alpha^2)(\nu^*)^2 z^*$$

$$l^*(\gamma^*)^2 = l^* \underbrace{\gamma^*}_{>(1-\alpha^2)z^*} \gamma^* > (1-\alpha^2)l^* z^*\gamma^* = (1+\alpha)(1-\alpha)l^* z^*\gamma^* > \alpha(1-\alpha)l^* z^*\gamma^*$$

$$
\begin{aligned}
& -(1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2 z^* \\
= \;& -(1-\alpha+\alpha)(1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2 z^* \\
= \;& -(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2 z^* - (1-\alpha^2)(1-\alpha)(\nu^*)^2 z^* \\
= \;& -(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2 z^* - (1-\alpha^2)(\nu^*)^2 z^* + \alpha(1-\alpha^2)\underbrace{\nu^*}_{>\frac{l^*}{1+\alpha}}\nu^* z^* \\
> \;& -(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2 z^* - (1-\alpha^2)(\nu^*)^2 z^* + \alpha(1-\alpha)l^*\nu^* z^*
\end{aligned}
$$

$$(1-\alpha^2)(1-\alpha)^2\nu^*(z^*)^2 = (1-\alpha^2)(1-\alpha)^2\nu^* \underbrace{z^*}_{>\frac{\nu^*}{\alpha}} z^* > (1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2 z^*$$

$$\frac{1-\alpha}{\alpha}\nu^*(\gamma^*)^2 = \frac{1-\alpha}{\alpha}\nu^* \underbrace{\gamma^*}_{>(1-\alpha^2)z^*} \gamma^* > (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^*\gamma^*.$$

Using these results and $\mathbf{x}^* > \mathbf{0}$, we find:

$$\overbrace{2l^*(\nu^*)^2}^{>0}\overbrace{+(1-\alpha)(1-2\alpha)l^*\nu^* z^*}^{>-\alpha(1-\alpha)l^*\nu^* z^*}\;\overbrace{+l^*\nu^*\gamma^*}^{>(1-\alpha^2)(\nu^*)^2 z^*}\;\overbrace{+\alpha^2(1-\alpha)^2 l^*(z^*)^2}^{>0}-\alpha(1-\alpha)l^* z^*\gamma^*\;\overbrace{+l^*(\gamma^*)^2}^{>\alpha(1-\alpha)l^* z^*\gamma^*}$$

$$
\begin{aligned}
&\underbrace{-(1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2 z^*}_{\substack{> -(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2 z^* \\ -(1-\alpha^2)(\nu^*)^2 z^* \\ +\alpha(1-\alpha) l^* \nu^* z^*}} \underbrace{+(1-\alpha^2)(1-\alpha)^2 \nu^* (z^*)^2}_{>(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2 z^*} -(1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* \gamma^* \underbrace{+\frac{1-\alpha}{\alpha}\nu^*(\gamma^*)^2}_{>(1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* \gamma^*} \\
> \quad & -\alpha(1-\alpha)l^*\nu^* z^* + (1-\alpha^2)(\nu^*)^2 z^* - \alpha(1-\alpha)l^* z^* \gamma^* + \alpha(1-\alpha) l^* z^* \gamma^* \\
& -(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2 z^* - (1-\alpha^2)(\nu^*)^2 z^* + \alpha(1-\alpha) l^* \nu^* z^* + (1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2 z^* \\
& -(1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* \gamma^* + (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* \gamma^* \\
= \quad & 0.
\end{aligned}
$$

Equation (22):

$\mathbf{b}_i e^{q_i t}$ are particular solutions to (10) ($\mathbf{b}_i = (b_{li}, b_{\nu i}, b_{zi}, b_{\gamma i})'$, $i = 1, \ldots, 4$). From $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x} - \mathbf{x}^*)$, it follows that $q_i \mathbf{b}_i e^{q_i t} = \mathbf{J}\mathbf{b}_i e^{q_i t}$, that is $(\mathbf{J} - q_i \mathbf{I})\mathbf{b}_i = \mathbf{0}$. The characteristic equation, $f(q) = |\mathbf{J} - q\mathbf{I}| = 0$, ensures that non-zero solutions $\mathbf{b}_i$ exist. $\mathbf{b}_i$ is the eigenvector corresponding to eigenvalue q_i ($i = 1, \ldots, 4$). The general solution of system (10) is $\mathbf{x}(t) - \mathbf{x}^* = \sum_{i=1}^4 B_i \mathbf{b}_i e^{q_i t}$, where the B_i's are constants to be determined below ($i = 1, \ldots, 4$). Since two eigenvalues, q_3 and q_4, say, are positive, we must have $\mathbf{x}(t) - \mathbf{x}^* = \sum_{i=1}^2 B_i \mathbf{b}_i e^{q_i t}$. Evaluating this equation at $t = 0$ yields (20):

$$
\begin{aligned}
\mathbf{x}(0) - \mathbf{x}^* &= \sum_{i=1}^{2} B_i \mathbf{b}_i \\
\begin{pmatrix} l(0) - l^* \\ \nu(0) - \nu^* \\ z(0) - z^* \\ \gamma(0) - \gamma^* \end{pmatrix} &= B_1 \begin{pmatrix} b_{l1} \\ b_{\nu 1} \\ b_{z1} \\ b_{\gamma 1} \end{pmatrix} + B_2 \begin{pmatrix} b_{l2} \\ b_{\nu 2} \\ b_{z2} \\ b_{\gamma 2} \end{pmatrix}.
\end{aligned} \tag{A.10}
$$

From the first the line in (A.10),

$$
B_2 = \frac{l(0) - l^* - b_{l1}B_1}{b_{l2}}. \tag{A.11}
$$

Inserting this into the third line in (A.10) yields:

$$
\begin{aligned}
z(0) - z^* &= B_1 b_{z1} + \frac{l(0) - l^* - b_{l1}B_1}{b_{l2}} b_{z2} \\
&= \frac{b_{z2}}{b_{l2}}[l(0) - l^*] + B_1 \left(b_{z1} - \frac{b_{l1} b_{z2}}{b_{l2}} \right) \\
B_1 &= \frac{z(0) - z^* - \frac{b_{z2}}{b_{l2}}[l(0) - l^*]}{b_{z1} - \frac{b_{l1} b_{z2}}{b_{l2}}}.
\end{aligned} \tag{A.12}
$$

Using (A.11) and (A.12) in the second equation in (A.10) yields (22):

$$
\nu(0) - \nu^* = B_1 b_{\nu 1} + \frac{l(0) - l^* - b_{l1}B_1}{b_{l2}} b_{\nu 2}
$$

$$
\begin{aligned}
&= \frac{b_{\nu 2}}{b_{l2}}[l(0)-l^*] + B_1\left(b_{\nu 1} - \frac{b_{l1}b_{\nu 2}}{b_{l2}}\right) \\
&= \frac{b_{\nu 2}}{b_{l2}}[l(0)-l^*] + \frac{z(0)-z^* - \frac{b_{z2}}{b_{l2}}[l(0)-l^*]}{b_{z1} - \frac{b_{l1}b_{z2}}{b_{l2}}}\left(b_{\nu 1} - \frac{b_{l1}b_{\nu 2}}{b_{l2}}\right) \\
&= [l(0)-l^*]\left[\frac{b_{\nu 2}}{b_{l2}} - \frac{\frac{b_{z2}}{b_{l2}}}{b_{z_1} - \frac{b_{l1}b_{z2}}{b_{l2}}}\left(b_{\nu 1} - \frac{b_{l1}b_{\nu 2}}{b_{l2}}\right)\right] + [z(0)-z^*]\frac{b_{\nu 1} - \frac{b_{l1}b_{\nu 2}}{b_{l2}}}{b_{z_1} - \frac{b_{l1}b_{z2}}{b_{l2}}} \\
&= [l(0)-l^*]\left[\frac{b_{\nu 2}}{b_{l2}} - \frac{\frac{b_{z2}}{b_{l2}}(b_{\nu 1}b_{l2} - b_{l1}b_{\nu 2})}{b_{z1}b_{l2} - b_{l1}b_{z2}}\right] + [z(0)-z^*]\frac{b_{\nu 1}b_{l2} - b_{l1}b_{\nu 2}}{b_{z1}b_{l2} - b_{l1}b_{z2}} \\
&= [l(0)-l^*]\frac{\frac{b_{\nu 2}}{b_{l2}}(b_{z1}b_{l2} - b_{l1}b_{z2}) - \frac{b_{z2}}{b_{l2}}(b_{\nu 1}b_{l2} - b_{l1}b_{\nu 2})}{b_{z1}b_{l2} - b_{l1}b_{z2}} + [z(0)-z^*]\frac{b_{\nu 1}b_{l2} - b_{l1}b_{\nu 2}}{b_{z1}b_{l2} - b_{l1}b_{z2}} \\
&= [l(0)-l^*]\frac{b_{z1}b_{\nu 2} - b_{\nu 1}b_{z2}}{b_{z1}b_{l2} - b_{l1}b_{z2}} + [z(0)-z^*]\frac{b_{\nu 1}b_{l2} - b_{l1}b_{\nu 2}}{b_{z1}b_{l2} - b_{l1}b_{z2}} \\
&= \frac{[l(0)-l^*](b_{z1}b_{\nu 2} - b_{\nu 1}b_{z2}) + [z(0)-z^*](b_{\nu 1}b_{l2} - b_{l1}b_{\nu 2})}{b_{z1}b_{l2} - b_{l1}b_{z2}}.
\end{aligned}
$$